

Asymmetric Auctions with Resale*

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Abstract

We study equilibria of first- and second-price auctions with resale in a model with independent private values. With asymmetric bidders, the resulting inefficiencies create a motive for post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser after updating his prior beliefs based on his winning. We show that a first-price auction with resale has a unique monotonic equilibrium. Our main result is that with resale, the expected revenue from a first-price auction exceeds that from a second-price auction. The results extend to other resale mechanisms: monopsony and, more generally, probabilistic k -double auctions. The inclusion of resale possibilities thus permits a general revenue ranking of the two auctions that is not available when these are excluded.

1 Introduction

In a first-price auction, asymmetries among bidders typically result in inefficient allocations—that is, the winner of the auction may not be the person who values the object the most. This inefficiency creates a motive for post-auction resale and when bidders take resale possibilities into account, their bidding behavior is affected. Standard models of such auctions, by and large, implicitly assume either that resale possibilities do not exist or that bidders do not take these into account when formulating bids.

There are at least two reasons why resale possibilities should be considered explicitly. The first one is positive. If, after the auction is over, bidders see that there are potential gains from trade, then they will naturally engage in such trade. And it seems unlikely that the seller can prevent bidders from engaging in post-auction

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trade, even if, for some reason, resale was deemed disadvantageous. In the auction of 3G spectrum licenses in the UK, post-auction trade was restricted by the government. The bidders, however, were easily able to circumvent these restrictions. TIW, a Canadian company, bid successfully for the most valuable license “A” with a winning bid in excess of £4 bn. A rival company, Hutchison, then acquired the license by buying TIW itself. British Telecom created a wholly-owned subsidiary that bid in the auction and also purchased a license. After the auction, this subsidiary was floated on the stock market and sold. Thus, restrictions on the buying and selling of licenses were circumvented by the buying and selling of companies that owned the licenses. The actions of British Telecom—it created the subsidiary before the auction—suggest that bidders fully anticipated post-auction resale possibilities.¹

The second reason to consider auctions with resale is normative. There has been recent interest in the design of efficient auctions, especially in the context of privatization. If post-auction resale results in efficiency, however, then from the planner’s perspective, an inefficient auction is just as good. Are the allocations from an inefficient auction followed by post-auction resale indeed efficient?

In this paper, we study the effects of post-auction resale in a simple model with two bidders whose private values are independently, but perhaps asymmetrically, distributed. Equilibrium allocations in first-price auctions are then inefficient and bidders have the incentive to engage in post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser.

We show that a first-price auction with resale has a unique monotonic equilibrium (Theorems 1 and 2). We do this by first showing that every equilibrium has the feature that, despite the asymmetries, the distributions of bids of the two bidders are identical. Symmetry would not be surprising if resale took place under complete information and so was always efficient. In that case, each bidder would bid as if his value were the maximum of the two values. In our model, the symmetry of bid distributions is striking because post-auction resale also takes place under incomplete information and so is not always efficient. Here it occurs as a result of some cost-benefit calculations at the margin. The symmetry of the bid distributions is key—it is used both to construct an equilibrium and to show that it is unique.

The possibility of resale also affects incentives in second-price auctions. It is no longer a dominant strategy to bid one’s value. It is, nevertheless, an *ex post* equilibrium and uniquely so. In this equilibrium, of course, the auction allocates efficiently and so there is no resale.

Our main result (Theorem 3) is that once resale possibilities are admitted, the expected revenue from a first-price auction exceeds that from a second-price auction. We thus obtain a *general* revenue ranking of the two auction formats. We require only that the value distributions be regular in the sense of Myerson (1981), ensuring that the monopoly pricing problem at the resale stage is well behaved. In particular,

¹We are grateful to Tilman Börgers for providing these details.

we do not assume that one of the bidders is “stronger” than the other in the sense of stochastic dominance. We remind the reader that in the absence of resale, the two auctions cannot be unambiguously ranked, even if the bidders can be classified as being “weak” and “strong” (Maskin and Riley (2000)).

The proof of Theorem 3 uses a perturbation technique borrowed from the calculus of variations. To the best of our knowledge, the use of this technique is new to auction theory and it will, perhaps, prove useful in other applications as well.

The results reported above concern a particular resale institution—monopoly pricing—in which the winner of the auction has all the bargaining power. In Section 6, we show, however, that this is inessential by first considering the monopsony mechanism in which the loser has all the bargaining power and then more generally, mechanisms in which bargaining power is shared, perhaps unequally. All of the results reported above extend to these alternative, and more general, resale institutions.

Related Literature Equilibrium analysis of asymmetric first-price auctions has posed many challenges. Some of the difficulties were already pointed out by Vickrey (1961) in his pioneering paper. He constructed an example in which bidder 1’s value, say a , was commonly known while the other’s was uniformly distributed. In that case, there is an equilibrium of the first-price auction in which bidder 1 randomizes. Vickrey (1961) showed that for some values of a , the revenue from a first-price auction exceeded that from a second-price auction; for other values of a , the revenue ranking was reversed.

Since then, progress in the area has been sporadic at best. In asymmetric first-price auctions without resale, pure strategy equilibria exist under quite permissive conditions, as a consequence of general existence results (see, for instance, Reny (1999), Athey (2001) or Jackson and Swinkels (2005)). There is, again under weak conditions, a unique equilibrium (see, for instance, Maskin and Riley (2003) or Lebrun (2006)). But characterization results and revenue comparisons are few and far between. Griesmer, et al (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Plum (1992) extends this to situations in which the two value distributions are of the form x^n , again over different supports. Cheng (2005) identifies a class of distribution pairs which lead to linear bidding strategies. For this class, he shows that the first-price auction is revenue superior to the second-price auction. For specific examples of distribution pairs, Cantillon (2005) shows how asymmetry affects revenue in first-price auctions. In the absence of general analytic results, some researchers have resorted to numerical methods (Marshall, et al (1994)).

Maskin and Riley (2000) derive the most comprehensive characterization and revenue ranking results concerning first- and second-price auctions in the presence of asymmetries. They consider problems in which one of the bidders is unambiguously stronger than the other. Specifically, the distribution of one bidder (conditionally) stochastically dominates that of the other. Maskin and Riley (2000) are able to

identify circumstances in which one or the other auction is revenue superior. For instance, the second-price auction is revenue superior if the distribution of the weak bidder is obtained from that of the strong bidder by reassigning probability mass toward lower values.

Gupta and Lebrun (1999) consider resale possibilities in a manner not unlike this paper. They assume, however, that at the end of the auction both *values* are announced. This means, of course, that resale is always efficient. But it is not clear how the auctioneer would come to know the values themselves. In contrast, in our model, the auctioneer knows only the bids and not the values. Haile (2003) considers resale possibilities in a *symmetric* model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed to them only after the auction. There is a motive for resale because although the winner of the auction may have received the highest signal, he may not have the highest true value. No general revenue ranking obtains. Zheng (2002) identifies conditions under which the outcomes of Myerson’s (1981) optimal auction can be achieved when resale is permitted.

The model of Garrat and Tröger (2006) is closest to ours in spirit. The crucial difference is that they assume, as in Vickrey’s (1961) example, that the value of one of the bidders is commonly known, and moreover, is equal to zero. This bidder, thus, participates in the auction for purely “speculative” reasons—he has no use value for the object. He only benefits if he can resell the object to the other bidder. In the efficient equilibrium of the second-price auction, the revenue is obviously zero. Garrat and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive. We allow for general continuous distributions and so their model may be viewed as a limiting case of ours.

2 Preliminaries

A single indivisible object is for sale. There are two risk-neutral buyers, labelled 1 and 2, with independently distributed private values, X_1 and X_2 . Buyer i ’s value for the object, X_i , is distributed according to the cumulative distribution function F_i with support $[0, \omega_i]$. It is assumed that each F_i admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, \omega_i)$. We suppose, without loss of generality, that $\omega_1 \geq \omega_2$.

We assume that both F_i are *regular* in the sense of Myerson (1981) so that for $i = 1, 2$, the *virtual value*, defined as

$$x - \frac{1 - F_i(x)}{f_i(x)}$$

is a strictly increasing function of the actual value x . This ensures that the price at the resale stage is uniquely determined and is characterized by the first-order conditions

for a maximum.²

In later sections we will need to consider conditional distributions of the form $F_i(x | X_i \leq a) = F_i(x) / F_i(a)$ with support $[0, a]$. The associated *conditional virtual value* is

$$x - \frac{1 - F_i(x | X_i \leq a)}{f_i(x | X_i \leq a)} = x - \frac{F_i(a) - F_i(x)}{f_i(x)}$$

The monotonicity of the virtual values implies the monotonicity of the conditional virtual values. To see this, note that if we write $q = F_i(a) \leq 1$, then the latter is equivalent to the statement: if $x < x' \leq a$ then

$$f_i(x')f_i(x)(x' - x) > f_i(x)[q - F_i(x')] - f_i(x')[q - F_i(x)]$$

If $f_i(x) < f_i(x')$, then the right-hand side is nonpositive for all q while the left-hand side is positive. If $f_i(x) \geq f_i(x')$, then the right-hand side is a nondecreasing function of q and since regularity implies that the inequality holds for $q = 1$, it holds for all $q < 1$. Thus, we have that for all $a < \omega_i$, the conditional virtual values are strictly increasing. In other words, if F_i is regular then the conditional distribution $F_i(\cdot | X_i \leq a)$ is also regular.

3 First-Price Auction with Resale

Our model of the first-price auction with resale (FPAR) is the following. The buyers first participate in a standard sealed-bid first-price auction. The winning bid is publicly announced. We assume—as is common in real-world auctions—that the losing bid is *not* announced.³

In the second stage, the winner of the auction—say j —may, if he wishes to, offer to sell the object to the other bidder $i \neq j$ at some price p . If the offer is accepted by i , a sale ensues. If the offer is rejected, the original owner j retains the object. Thus resale takes place via a take-it-or-leave-it offer by the winner of the auction.⁴

Strategies and Beliefs A *strategy* for bidder i in a first-price auction with resale has two components: (i) a *bidding strategy* $\beta_i : [0, \omega_i] \rightarrow \mathbb{R}$ that specifies the bid $\beta_i(x_i)$ that i will submit in the auction if his value is x_i ; (ii) a *pricing strategy* $p_i(b_i, x_i)$ which specifies the asking price i will set if he wins the auction with a bid of

²As shown by Bulow and Roberts (1989), the virtual value can be interpreted as the “marginal revenue” of a monopolist who faces a demand curve $1 - F_i(p)$.

³This assumption is discussed in more detail below in Remark 1.

⁴All bargaining power thus lies with the seller and from his perspective, this is, of course, the optimal resale mechanism. In Section 6 below, we show that our analysis extends to resale mechanisms in which all bargaining power lies with the buyer and then, more generally, to mechanisms in which it is shared.

b_i and his value is x_i . We allow i to set $p_i = \infty$, which is interpreted to be the same as a decision not to offer the object for sale.

In addition, i must specify the set of price offers $A_i(b_j, x_i)$ he will accept if j wins with a bid of b_j and his own value is x_i .

A *belief function* μ_i specifies a probability distribution $\mu_i(b_i, \cdot)$ over $[0, \omega_j]$ that represents the beliefs that i holds regarding j 's values if he wins the auction with a bid of b_i .

Equilibrium A *perfect Bayesian equilibrium* (henceforth, *equilibrium*) consists of a pair of bidding-pricing strategies (β_i, p_i) and belief functions μ_i for $i = 1, 2$ with the property that: (i) if i loses the auction, then the set of offers he will accept is $A_i(b_j, x_i) = \{p_j : p_j \leq x_i\}$; (ii) if i wins the auction with a bid of b_i when his value is x_i , then $p_i(b_i, x_i)$ is optimal given $\mu_i(b_i, \cdot)$ and the acceptance strategy above; (iii) for each x_i , $\beta_i(x_i)$ is optimal given β_j, p_i and p_j ; and (iv) the beliefs μ_i are generated from F_i and β_i using Bayes rule whenever possible.

Note that if i loses the auction, then the announcement of the winning bid b_j carries no useful information—that is, the set of price offers i will accept is independent of b_j . Thus the equilibrium would be unaffected if neither bid were announced.

As usual, we work backwards and first outline behavior in the resale stage.

3.1 Resale Stage

Suppose that the two bidders follow *continuous* and *strictly increasing* bidding strategies β_1 and β_2 with inverses ϕ_1 and ϕ_2 , respectively.⁵

Suppose that bidder j with value x_j won the auction with a bid of b . As a result, he would infer that bidder i 's value $X_i \leq \phi_i(b)$. His beliefs, therefore, are $\mu_j(b, x_i) = F_i(x_i | X_i \leq \phi_i(b)) = F_i(x_i) / F_i(\phi_i(b))$. If $x_j < \phi_i(b)$, then there are potential gains from trade and so bidder j will set a (“monopoly”) price p that maximizes

$$[F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j$$

The first term in the maximand is j 's expected payoff from the event $X_i \geq p$ in which bidder i accepts his offer. The second term is his payoff from the event $X_i < p$, in which case bidder i rejects it.

The first-order condition for j 's maximization problem can be rewritten as

$$p - \frac{F_i(\phi_i(b)) - F_i(p)}{f_i(p)} = x_j \tag{1}$$

Since F_i is regular, the left-hand side is increasing and so (1) has a unique solution. Moreover, (1) is also sufficient for j 's maximization problem. Thus there is a unique price

$$p_j(b, x_j) \equiv \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j \tag{2}$$

⁵We will show later that all equilibria must have these properties.

that maximizes j 's payoff from resale and clearly, $x_j < p_j(b, x_j) < \phi_i(b)$. It follows immediately from (1) that the optimal price $p_j(b, x_j)$ is an increasing function of both b and x_j .

Let

$$R_j(b, x_j) = \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j \quad (3)$$

denote bidder j 's optimal expected revenue from resale. For future reference, note that as a result of the envelope theorem,

$$\frac{\partial}{\partial b} R_j(b, x_j) = f_i(\phi_i(b)) \phi_i'(b) p_j(b, x_j) \quad (4)$$

If bidder j won the auction with a bid of b and $x_j \geq \phi_i(b)$, then there are no potential gains from trade and so bidder j does not offer the object for sale.

3.2 Bidding Stage

We begin by deriving some necessary conditions that equilibrium bidding strategies must satisfy. At the time of the auction, both bidders anticipate that behavior in the resale stage will be as specified above.

3.2.1 Necessary Conditions

Suppose that, in equilibrium, each bidder i follows a *continuous* and *strictly increasing* bidding strategy $\beta_i : [0, \omega_i] \rightarrow \mathbb{R}$, so that $\beta_i(x_i)$ is the bid submitted by i when his value is x_i .

First, notice that we must have $\beta_1(0) = \beta_2(0) = 0$. Suppose that $\beta_i(0) > 0$ and without loss of generality, suppose that for some small δ , it is the case that for all $x \in (0, \delta)$, $\beta_i(x) \geq \beta_j(x)$. For $x_i \in (0, \delta)$ small enough, bidder i 's value is less than $\beta_i(0)$ and if he wins and resells to bidder j , the price he will receive is also less than $\beta_i(0)$. This is because if he wins with bid of $\beta_i(x_i)$ then j 's value $X_j \leq \beta_j^{-1}\beta_i(x_i)$ and by continuity this is close to zero, when x_i is close to zero. Thus his gain from winning is less than $\beta_i(0)$ whether or not the object is resold. Moreover, if bidder i with value x_i loses, then $x_i \leq X_j$ and there will be no resale. Thus his payoff from losing is 0. As a result the overall payoff is negative and it is better bidder i with value x_i to bid 0. Hence, $\beta_i(0) > 0$ is not possible.

It is also easy to verify that $\beta_1(\omega_1) = \beta_2(\omega_2) \equiv \bar{b}$.

As above, let $\phi_i : [0, \bar{b}] \rightarrow [0, \omega_i]$ denote i 's inverse bidding strategy in equilibrium, that is, $\phi_i = \beta_i^{-1}$. Fix a bid b and suppose that $\phi_j(b) < \phi_i(b)$. This means that if j wins with a bid of b , then there are potential gains from trade and so j will make an offer to i . If, on the other hand, i wins with bid of b , then there are no potential gains from trade and so i will not make an offer to j .

Suppose bidder i follows ϕ_i . Bidder j 's expected payoff when his value is $x_j \equiv \phi_j(b)$ and he deviates by bidding a c close to b is

$$\Pi_j(c, x_j) = R_j(c, x_j) - F_i(\phi_i(c))c$$

where $R_j(c, x_j)$, defined in (3), is his expected payoff from resale if he wins the auction. If j loses the auction, then $\phi_j(c) < \phi_i(c)$ implies that bidder i will not offer to resell to him and so his payoff is 0. Since it is optimal for j to bid b , the first-order condition for maximizing Π_j , together with (4), results in

$$f_i(\phi_i(b))\phi_i'(b)p_j(b, x_j) - f_i(\phi_i(b))\phi_i'(b)b - F_i(\phi_i(b)) = 0$$

where $p_j(b, x_j)$ is defined in (2). Since $x_j = \phi_j(b)$, writing $p(b) \equiv p_j(b, \phi_j(b))$, the first-order condition results in the differential equation

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{p(b) - b} \quad (5)$$

Note that p depends on both ϕ_1 and ϕ_2 .

Now suppose bidder j follows an equilibrium strategy ϕ_j . Bidder i 's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\Pi_i(c, x_i) = (x_i - c)F_j(\phi_j(c)) + \int_{\phi_j(c)}^{\omega_j} \max\{[x_i - p_j(\beta_j(x_j), x_j)], 0\} f_j(x_j) dx_j$$

This is because if i wins the auction, he never resells to j and so his profit is simply $x_i - c$. The second term is i 's expected payoff from buying the object from j . Since it is optimal for i to bid b , the first-order condition for maximizing Π_i is

$$[x_i - b] f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) - [x_i - p_j(b, \phi_j(b))] f_j(\phi_j(b))\phi_j'(b) = 0$$

Again writing $p_j(b, \phi_j(b)) = p(b)$, the first-order condition becomes

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{p(b) - b}$$

which is the *same* as (5).

We have argued that if ϕ_1, ϕ_2 are the equilibrium inverse bid functions in a first-price auction with resale, then they must both satisfy (5) for all b . This was derived using the first-order necessary conditions for local deviations to be unprofitable.

3.2.2 Sufficiency

We now show that solutions to the differential equations (5) for $j = 1, 2$ are indeed equilibrium bidding strategies—that is, no deviations are profitable.

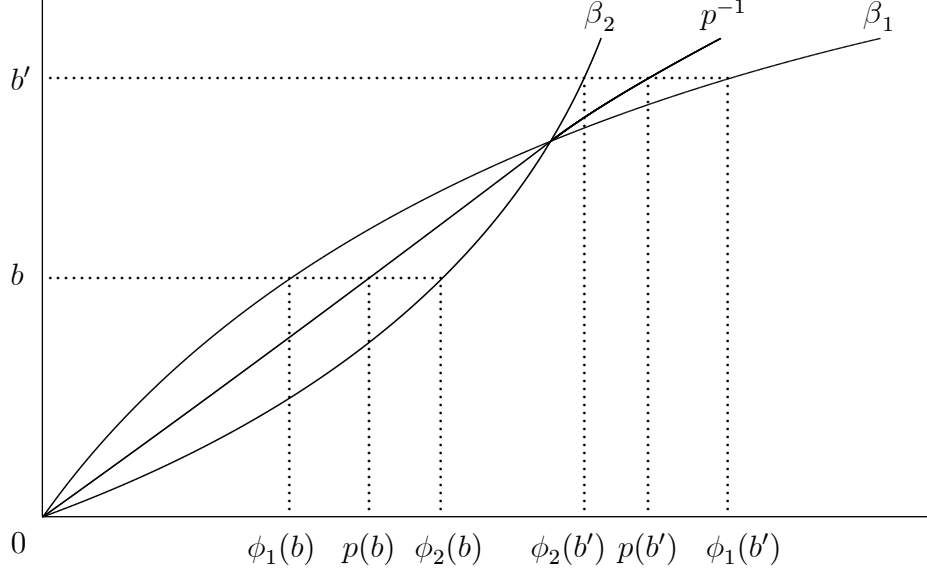


Figure 1: Bidding and Pricing Functions

Proposition 1 *The strictly increasing and onto functions $\phi_1 : [0, \bar{b}] \rightarrow [0, \omega_1]$ and $\phi_2 : [0, \bar{b}] \rightarrow [0, \omega_2]$ are equilibrium inverse bidding strategies for the first-price auction with resale if and only if for all $b \in [0, \bar{b}]$,*

$$\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{p(b) - b} \quad (6)$$

$$\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{p(b) - b} \quad (7)$$

where, if $\phi_j(b) \leq \phi_i(b)$,

$$p(b) = \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j(b) \quad (8)$$

Proof. See Appendix A. ■

Note that the boundary conditions are determined by the condition that the ϕ_i be strictly increasing and onto.

A word regarding the *equilibrium pricing function* $p(\cdot)$ and its relationship to the equilibrium inverse bidding strategies, ϕ_1 and ϕ_2 , is in order. Suppose, as depicted in Figure 1, that $\phi_1(b) < \phi_2(b)$. Then if bidder 1 were to win with a bid of b , he would infer that $X_2 \leq \phi_2(b)$ and since there are potential gains from trade, bidder 1 would make an offer $p(b)$ to bidder 2 ($j = 1$ and $i = 2$ in formula (8)). On the other hand, if bidder 2 were to win with a bid of b , he would infer that 1's value $X_1 \leq \phi_1(b) < \phi_2(b) = x_2$ and so would not make 1 an offer. But for bid b' ,

$\phi_2(b') < \phi_1(b')$, again as in the figure, the opposite holds. Upon winning with a bid of b' , bidder 2 would make an offer $p(b')$ to bidder 1 ($j = 2$ and $i = 1$ in formula (8)) but not the other way around. The value of b thus uniquely determines the identity of the seller j who sets the price $p(b)$.

Recall that in any increasing equilibrium, the highest bids must be the same, say \bar{b} . Thus $F_1(\phi_1(\bar{b})) = 1 = F_2(\phi_2(\bar{b}))$. Since the boundary conditions for the two differential equations are the same, it now follows immediately from Proposition 1 that

Corollary 1 *If $\phi_1 : [0, \bar{b}] \rightarrow [0, \omega_1]$ and $\phi_2 : [0, \bar{b}] \rightarrow [0, \omega_2]$ are strictly increasing equilibrium inverse bidding strategies, then for all $b \in [0, \bar{b}]$,*

$$F_1(\phi_1(b)) = F_2(\phi_2(b))$$

that is, the bid distributions of the two bidders are identical.

The equality of the bid distributions implies that for all $x \in [0, \omega_2]$, $F_i(x) \leq F_j(x)$ if and only if $\beta_i(x) \leq \beta_j(x)$. In particular, the bidding functions intersect at x if and only if the distributions intersect at x .

3.2.3 Symmetrization

Corollary 1 identifies a remarkable property of first-price auctions with resale—even though the bidders are asymmetric, in equilibrium they bid in a way that the resulting bid distributions $F_i(\phi_i(\cdot))$ are the same. In this sense, resale *symmetrizes* the auction. Since this property plays an important role in what follows, it is worth exploring the underlying reasons.

As a first step, consider a standard first-price auction *without resale* (FPA) and let φ_1 and φ_2 be the equilibrium inverse bidding strategies. Suppose bidder i with value $x_i = \varphi_i(b)$ raises his bid slightly from b to $b + \varepsilon$. By doing this bidder i gains approximately $x_i - b = \varphi_i(b) - b$ whenever $\varphi_j(b) < x_j < \varphi_j(b + \varepsilon)$. These are the types of bidder j against whom bidder i loses the auction by bidding b but wins by bidding $b + \varepsilon$. Writing the first-order condition for optimality yields the pair of differential equations: for $j = 1, 2$ and $i \neq j$,

$$\frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b} \tag{9}$$

Notice that the right-hand side is the inverse of the marginal gain accruing to i from increasing his bid.

Now consider a first-price auction *with resale* (FPAR) with equilibrium inverse bidding strategies ϕ_1 and ϕ_2 . Let b be such that $\phi_j(b) < \phi_i(b)$. This means that in equilibrium if j wins with a bid of b , so that his value $x_j = \phi_j(b)$, then he will try to

resell the object to bidder i since there are potential gains from trade. On the other hand, if i wins with a bid of b , he will not resell the object to bidder j since there are no gains from trade.

Suppose bidder j with value $x_j = \phi_j(b)$ raises his bid slightly from b to $b + \varepsilon$. As before, we look at how much j gains against bidder i types such that $\phi_i(b) < x_i < \phi_i(b + \varepsilon)$. When he bids b , bidder j loses against these types of bidder i and since there is no resale, bidder j 's payoff is 0. When he bids $b + \varepsilon$, however, he wins against these types of bidder i and is able to resell to them at a price of $p(b)$ for a gain of $p(b) - b$.

What about bidder i ? Suppose bidder i with value $x_i = \phi_i(b)$ raises his bid slightly from b to $b + \varepsilon$ and again consider the benefit to i against those bidder j types such that $\phi_j(b) < x_j < \phi_j(b + \varepsilon)$. When he bids b , bidder i loses against these types of bidder j but is able to buy the object from them at a price of approximately $p(b)$. His payoff thus approximately equals $x_i - p(b)$. When he bids $b + \varepsilon$, he wins against these types of bidder j and so his payoff is $x_i - b$. The gain in payoff for i from increasing his bid from b to $b + \varepsilon$ is thus approximately equal to $(x_i - b) - (x_i - p(b)) = p(b) - b$, the same as i 's gain!

The symmetrization effects of resale come from the fact that the *marginal gain* to both bidders from a higher bid is the same: $p(b) - b$. In contrast to (9), the right-hand sides of (6) and (7) are identical.

The distributions of equilibrium bids in an asymmetric first-price auction with resale are thus observationally equivalent to the distribution of equilibrium bids in a symmetric first-price auction. In other words, given F_1 and F_2 , there exists a distribution F such that a first-price auction (FPA) in which both bidders draw values from F is equivalent, in terms of equilibrium bid distributions, to a first-price auction with resale (FPAR) in which bidders draw values from F_1 and F_2 , respectively. This also means that the two auctions are revenue equivalent.

We now show how F may be obtained from F_1 and F_2 .

Lemma 1 *Given distributions F_1 and F_2 , define F as follows: if $F_i(p) \leq F_j(p)$, then*

$$F(p) = F_j \left(p - \frac{F(p) - F_i(p)}{f_i(p)} \right) \quad (10)$$

Then F is a uniquely determined distribution function such that $F_i(p) \leq F(p) \leq F_j(p)$. Moreover, if $F_i(p) < F_j(p)$, then $F_i(p) < F(p) < F_j(p)$.

Proof. See Appendix A. ■

The construction of F has a simple geometric interpretation, as depicted in Figure 2. For the point p , $F_2(p) < F_1(p)$ and so $i = 2$ and $j = 1$ in the formula (10). The distribution F is such that it passes through the point b , which bisects the line segment ac . The length of the line segment ab is just $p - F_1^{-1}(F(p))$. And since bd/bc is $f_2(p)$, the slope of F_2 at p , the length of bc is $[F(p) - F_2(p)] / f_2(p)$. Equation (10) requires that these be equal.

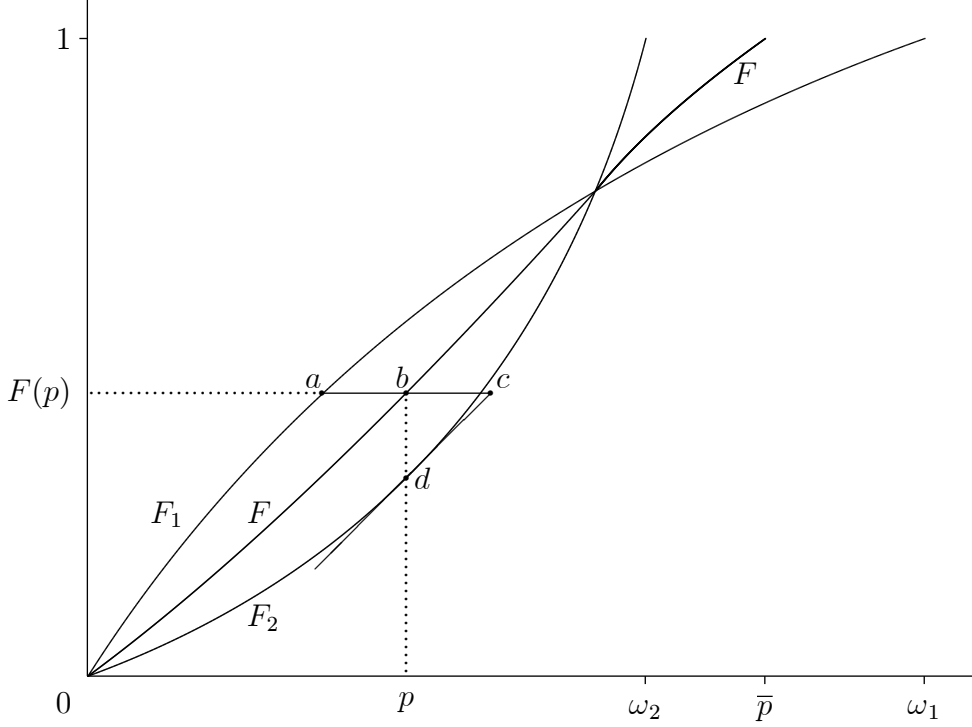


Figure 2: Construction of F

3.2.4 Equivalent Symmetric Auction

Now consider a *symmetric* first-price auction without resale in which there are two bidders and both draw values independently from the distribution function F on $[0, \bar{p}]$ as defined above in (10).

The equilibrium strategies in a symmetric auction can, of course, be derived explicitly and are given by

$$\beta(x) = \frac{1}{F(x)} \int_0^x y f(y) dy$$

Let $\bar{b} = \beta(\bar{p})$. Define the equilibrium inverse bid function for the symmetric auction as

$$\phi(b) \equiv \beta^{-1}(b) \tag{11}$$

so that the distribution of bids for each bidder is $F(\phi(b))$. A necessary condition for ϕ to be the equilibrium inverse bidding strategy in the symmetric auction is that

$$\frac{d}{db} \ln F(\phi(b)) = \frac{1}{\phi(b) - b} \tag{12}$$

3.3 Existence and Uniqueness of Equilibrium

In this section, we establish that the first-price auction with resale has a pure strategy equilibrium in which each bidder follows a strictly increasing bidding strategy. The equilibrium is unique in the class of pure strategy equilibria with nondecreasing bidding strategies.

The proof that there is a strictly increasing equilibrium is constructive. Given regular distribution functions F_1 and F_2 , construct F as in Lemma 1. Consider a symmetric first-price auction in which each bidder draws values independently from F . In symmetric auctions, it is known that a symmetric equilibrium β exists and is strictly increasing. We will use the equilibrium β to construct equilibrium bidding strategies β_1 and β_2 for the asymmetric first-price auction with resale.

3.3.1 Existence of Equilibrium

Theorem 1 *Suppose F_1 and F_2 are regular. Then there exists an equilibrium in the first-price auction with resale in which the bidding strategies are strictly increasing.*

Proof. The proof is by construction.

Given F_1 and F_2 , let F be determined as in the statement of Lemma 1. Let ϕ , as defined above in (11), be the equilibrium inverse bidding strategy in the symmetric auction in which bidders draw values from F . Let \bar{b} be the maximum bid in the symmetric auction and define inverse bidding strategies $\phi_1 : [0, \bar{b}] \rightarrow [0, \omega_1]$ and $\phi_2 : [0, \bar{b}] \rightarrow [0, \omega_2]$ in the asymmetric first-price with resale as follows:

$$F_1(\phi_1(b)) = F(\phi(b)) \quad (13)$$

$$F_2(\phi_2(b)) = F(\phi(b)) \quad (14)$$

Then using (12), we have that for $i = 1, 2$

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{\phi(b) - b}$$

We claim that ϕ_1 and ϕ_2 are equilibrium inverse bidding strategies in the first-price auction with resale.

The definition of F in (10) implies that if $F_i(\phi(b)) < F_j(\phi(b))$, then

$$F_j^{-1}(F(\phi(b))) = \phi(b) - \frac{F(\phi(b)) - F_i(\phi(b))}{f_i(\phi(b))}$$

and since $F(\phi(b)) = F_j(\phi_j(b)) = F_i(\phi_i(b))$,

$$\phi_j(b) = \phi(b) - \frac{F_i(\phi_i(b)) - F_i(\phi(b))}{f_i(\phi(b))}$$

which is precisely the first-order condition for

$$p(b) = \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j(b)$$

Regularity implies that the first-order condition is both necessary and sufficient for a maximum. Thus we have that for all b

$$p(b) = \phi(b)$$

Finally, note that $F_i(p(b)) < F_j(p(b))$ is equivalent to $\phi_j(b) < \phi_i(b)$. This is because (13) and (14) imply that $F_i(p(b)) < F(p(b))$, which is equivalent to $F_i(p(b)) < F_i(\phi_i(b))$ and so also to $p(b) < \phi_i(b)$. Similarly, $F(p(b)) < F_j(p(b))$ is equivalent to $F_j(\phi_j(b)) < F_j(p(b))$ and so also to $\phi_j(b) < p(b)$. Thus, $F_i(p(b)) < F_j(p(b))$ if and only if $\phi_j(b) < \phi_i(b)$.

We have thus argued that if ϕ_1 and ϕ_2 are determined by (13) and (14), then they satisfy the differential equations (6) and (7) where $p(b)$ is determined by (8).

Thus as constructed, the functions ϕ_1 and ϕ_2 satisfy the conditions of Proposition 1 and so constitute equilibrium inverse bidding strategies. ■

Remark 1 Theorem 1 relies on the assumption that at the end of the auction, the losing bid is not announced. If the losing bid is announced, the value of the losing bidder would be revealed in any strictly increasing equilibrium. This creates an incentive for a bidder to bid lower, so that if he were to lose, then the other bidder would think that his value is smaller than it actually is. This effect overwhelms the loss from not winning with a lower bid and it can be shown that no strictly increasing equilibrium exists. In fact, a stronger result holds: if the losing bid is announced, there is no nondecreasing equilibrium with (partial) pooling either. A formal proof of this is available from the authors.

3.3.2 Uniqueness of Equilibrium

In this section we show that the equilibrium constructed in Theorem 1 is, in fact, the only equilibrium in which bidders follow *nondecreasing* bidding strategies. We first show that it is unique in the class of equilibria with strictly increasing strategies. The proof is completed by showing that if both equilibrium bidding strategies are nondecreasing, then they must be strictly increasing and continuous.

The following proposition is a key step. It shows that the distribution of resale prices in any equilibrium is given by F . Since F is determined without reference to the equilibrium, this shows that the distribution of resale prices is uniquely determined.

Proposition 2 *Suppose ϕ_1 and ϕ_2 are strictly increasing equilibrium inverse bidding strategies. Then F , defined in (10), is the distribution of equilibrium resale prices and for all b , $F(p(b)) = F_j(\phi_j(b))$, $j = 1, 2$.*

Proof. Let the random variable P denote the resale price resulting from ϕ_1 and ϕ_2 . Fix a b such that $\phi_j(b) < \phi_i(b)$ and suppose bidder j wins the auction with a bid of b . Then we have.

$$\begin{aligned}\Pr [P \leq p(b)] &= \Pr [X_j \leq \phi_j(b)] \\ &= F_j(\phi_j(b))\end{aligned}$$

Because in equilibrium, for all b , $F_1(\phi_1(b)) = F_2(\phi_2(b))$, it is also the case that $\Pr [P \leq p(b)] = F_i(\phi_i(b))$. And since $p(b)$ is the monopoly resale price when the winning bid is b , it must satisfy the first-order condition

$$\phi_j(b) = p(b) - \frac{F_i(\phi_i(b)) - F_i(p(b))}{f_i(p(b))}$$

So

$$\begin{aligned}\Pr [P \leq p(b)] &= F_j(\phi_j(b)) \\ &= F_j\left(p(b) - \frac{\Pr [P \leq p(b)] - F_i(p(b))}{f_i(p(b))}\right)\end{aligned}$$

where we have used the fact that $\Pr [P \leq p(b)] = F_i(\phi_i(b))$ also. Now (10) implies that

$$\Pr [P \leq p(b)] = F(p(b))$$

and this completes the proof. ■

Since $F(p(b)) = F_j(\phi_j(b))$, Proposition 1 implies that

$$\frac{d}{db}(F(p(b))) = \frac{1}{p(b) - b}$$

This means that $p(b)$ satisfies (12), the differential equation characterizing the equilibrium inverse bidding strategy in the symmetric first-price auction in which both bidders draw values independently from the same distribution F . But given F , the equilibrium inverse bidding strategy in a symmetric first-price auction is uniquely determined.

We have thus argued that given F_1 and F_2 , the equilibrium distribution of resale prices, F , and the function $p(b)$ are uniquely determined. Now it follows from $F_i(\phi_i(b)) = F(p(b))$ that ϕ_1 and ϕ_2 are uniquely determined. Thus there is only one equilibrium in which bidders follow *strictly increasing* bidding strategies.

In Appendix A it is shown that there are no equilibria with nondecreasing strategies that are not strictly increasing. Thus, we obtain

Theorem 2 *The first-price auction with resale has a unique equilibrium in the class of equilibria with nondecreasing bidding strategies.*

3.4 An Example

It is useful to consider an explicit example to illustrate the various constructs.

Example 1 *Suppose that $F_1(x) = x/\omega_1$ over $[0, \omega_1]$ and $F_2(x) = x/\omega_2$ over $[0, \omega_2]$ where $\omega_1 \geq \omega_2$; that is, the value distributions are both uniform, but over different supports.*

Since $F_1 \leq F_2$, it can only be that (the “weak”) bidder 2 resells to (the “strong”) bidder 1. It may be verified that, the distribution of resale prices, F , is also uniform. Specifically, $F(p) = 2p/(\omega_1 + \omega_2)$ over support $[0, \frac{1}{2}(\omega_1 + \omega_2)]$. The associated pricing function $p(b) = 2b$. The equilibrium inverse bidding strategies are: $\phi_1(b) = 4\omega_1 b/(\omega_1 + \omega_2)$ and $\phi_2(b) = 4\omega_2 b/(\omega_1 + \omega_2)$. The highest bid $\bar{b} = \frac{1}{4}(\omega_1 + \omega_2)$.

Notice that if $3\omega_2 < \omega_1$, then $\phi_2(b) < b$, or equivalently, $\beta_2(x) > x$; that is, bidder 2 bids more than his value in a first-price auction with resale. The reason, of course, is that he anticipates being able to resell the object to bidder 1 for a profit. Thus, bidder 2’s motives have a substantial “speculative” component. The model of Garrat and Tröger (2006), in which one of the bidders is known to have a value of 0, is an extreme instance of this. There speculation is the only motive.

3.5 Revenue

The auxiliary symmetric first-price auction in which both bidders draw values from F has the same distribution of bids as the first-price auction with resale in which bidders draw values from F_1 and F_2 , respectively. This is because $p(\cdot)$ is the equilibrium inverse bidding strategy in the auxiliary auction and from Lemma 2, for all $b \in [0, \bar{b}]$, $F(p(b)) = F_j(\phi_j(b))$.

Hence, in equilibrium, the expected revenue accruing to the auctioneer from a first-price auction with resale (FPAR) is

$$\begin{aligned} R^{FPAR}(F_1, F_2) &= R^{FPA}(F, F) \\ &= R^{SPA}(F, F) \\ &= \int_0^{\bar{p}} (1 - F(p))^2 dp \end{aligned} \tag{15}$$

where F is defined in (10) and $R^{SPA}(F, F)$ denotes the revenue from a symmetric second-price auction (SPA). The second equality is a consequence of the revenue equivalence principle. The third equality is a well-known formula for the expectation of the minimum of two independent random variables, both of which are distributed according to F .

Note that the expected revenue from a first-price auction with resale can thus be calculated without direct reference to the equilibrium bidding strategies.

For the asymmetric uniform distributions in Example 1, we have

$$R^{FPAR}(F_1, F_2) = \frac{1}{6}(\omega_1 + \omega_2)$$

4 Second-Price Auction with Resale

We now study properties of the second-price auction with resale. Our model is the same as that in previous sections except for the change in the auction format—that is, there is a second-price auction and then the winner, if he so wishes, can resell the object to the other bidder via a take-it-or-leave-it offer. There is one important difference, however. Under second-price rules, the winner of the auction inevitably knows the *losing* bid—after all this is the price he pays in the auction. Thus, unlike in a first-price auction, the winner can condition the price offered in the resale stage on the losing bid.⁶ This, of course, considerably simplifies the inference problem faced by a winning bidder and puts the losing bidder in a weak position during resale.

Resale Stage Suppose bidder i follows a nondecreasing bidding strategy β_i in the auction. Suppose also that bidder j wins the auction and pays a price of b_i ; that is, i 's bid. He can then infer that bidder i 's value is in the set $\beta_i^{-1}(b_i) = \{x : \beta_i(x) = b_i\}$.

If $\beta_i^{-1}(b_i)$ is a singleton, say $\beta_i^{-1}(b_i) = \{x_i\}$, then it is optimal for j to offer the object to i only if $x_j < x_i$ and in that case, set a price $p = x_i$. If $\beta_i^{-1}(b_i)$ is an interval, then it is optimal for j to offer the object to i only if $x_j < \sup \beta_i^{-1}(b_i)$ and in that case, set a price p that maximizes

$$[F_i(\sup \beta_i^{-1}(b_i)) - F_i(p)]p + [F_i(p) - F_i(\inf \beta_i^{-1}(b_i))]x_j$$

Bidding Stage With private values, a standard second-price auction—without the possibility of resale—has some important and well-known features. First, it is a *weakly dominant* strategy for each bidder to bid his true value. Second, the resulting equilibrium is, of course, *efficient*, even in an asymmetric environment. Third, there is a continuum of other (inefficient) equilibria (see Blume and Heidhues (2004) for a complete classification).

Our first observation is that once there is the possibility of resale, it is *not* a weakly dominant strategy to bid one's value in a second-price auction. As the example below shows, if one of the bidders, say 1, bids more than his value, the other bidder may gain by bidding less than his value. This is because a lower bid in the auction may lead to a lower resale price.

Example 2 *The values $X_1, X_2 \in [0, 1]$. Suppose that bidder 1 bids according to a continuous and strictly increasing strategy β_1 such that $\beta_1(x) > x$, for all $x \in (0, 1)$, and, if he wins, has beliefs $\mu_1(b_2, x) = 1$ if $x = b_2$ and 0 otherwise; that is, 1 believes that 2 is following the strategy $\beta_2(x_2) = x_2$.*

Suppose that bidder 1 has value $x_1 \in (0, 1)$ and bidder 2's value x_2 is such that $x_1 < x_2 < \beta_1(x_1)$. If bidder 2 bids x_2 , then 1 will win the auction and will offer the

⁶Recall that a first-price auction with resale does not have a monotonic equilibrium if the losing bid is known to the winner. See Remark 1.

object to 2 at price $p = x_2$. So bidder 2's payoff from bidding his value is 0. If bidder 2 reduces his bid to a b_2 such that $x_1 < b_2 < x_2$, then again bidder 1 will win the auction but now offer to sell the object to 2 at price $p = b_2$. By accepting this offer, bidder 2 can make a profit of $x_2 - b_2$. Thus in this situation it is strictly better for bidder 2 to bid $b_2 < x_2$ than to bid x_2 .

Ex Post Equilibrium While not weakly dominant, if both bidders bid their values, then this nevertheless results in an equilibrium of the second-price auction with resale. In fact, it constitutes an *ex post equilibrium*—that is, a perfect Bayesian equilibrium in which for (almost) all realizations of the values, x_1 and x_2 , neither bidder regrets his actions after learning the value of the other bidder. Equivalently, an ex post equilibrium is one in which the bidders' strategies do not depend on the distributions of the values. The property that such equilibria are “belief free” makes them particularly attractive. Of course, every weakly dominant equilibrium is an ex post equilibrium.

Proposition 3 *There is an ex post equilibrium of the second-price auction with resale in which both bidders bid their values.*

Proof. Suppose bidder i follows the bidding strategy $\beta_i(x) = x$.

Suppose bidder j deviates and bids $x < x_j$. If $x_i < x < x_j$, bidder j wins for a price of x_i and so there is no resale. So his payoff is $x_j - x_i$ which is the same as if he bid x_j . If $x < x_i < x_j$, then i wins and j 's payoff is zero since again there is no offer of resale. So if $x < x_i < x_j$, j 's payoff is zero if he bids x and $x_j - x_i$ if he bids x_j . Finally, if $x < x_j < x_i$, bidder j loses the auction and his payoff is 0 whether he bids x or x_j . Thus underbidding is not profitable.

Now suppose bidder j bids $x > x_j$. If $x_i > x$, then j 's payoff is 0 since he loses and i will not resell to him. If $x > x_i > x_j$, then again his payoff is zero, because he will pay x_i for the object and then resell to i for X_i . If $x > x_j > x_i$, then it makes no difference whether he bids x or x_j . Thus overbidding is not profitable either.

We have thus argued it is a best response for bidder j to follow the strategy $\beta_j(x) = x$, also.

Since the allocations at the end of the auction are efficient, there is no resale.

All of the arguments above are ex post—that is, they are valid even if j knows i 's value and so bidding one's value constitutes an ex post equilibrium. ■

Uniqueness In a second-price auction (SPA) without resale, it is an ex post equilibrium, of course, for each bidder to bid his value. This because clearly every dominant strategy equilibrium is an ex post equilibrium. But in a SPA there are also *other* ex post equilibria.

Example 3 The values $X_1, X_2 \in [0, 1]$. Bidder 1 bids according to the strategy $\beta_1(x) = 1$ and bidder 2 according to $\beta_2(x) = 0$.

These strategies, while weakly dominated, nevertheless constitute an ex post equilibrium. Thus in the second-price auction without resale, there is a multiplicity of ex post equilibria.

When there is resale, however, there is (essentially) a unique ex post equilibrium—both bidders bid their values and since the resulting allocation is efficient, there is no resale.

Proposition 4 Suppose β_1 and β_2 are bidding strategies in an ex post equilibrium of the second-price auction with resale. Then for all $x \in [0, \omega_2]$, $\beta_1(x) = \beta_2(x) = x$.

Proof. See Appendix B. ■

If $\omega_1 > \omega_2$, then there are ex post equilibria in which for $x > \omega_2$, bidder 1 bids more than x . Since bidder 1 wins for sure in these cases, all such equilibria are outcome equivalent to the one in which for all x , bidder 1 bids his value.

In what follows, we restrict attention to the unique ex post equilibrium outcome of the second-price auction with resale in which bidders bid their values and the outcome is efficient. Henceforth, we will refer to this as the *efficient* equilibrium.

Revenue The expected revenue from the efficient equilibrium of a second-price auction with resale (SPAR) is

$$R^{SPAR}(F_1, F_2) = \int_0^{\omega_2} (1 - F_1(x))(1 - F_2(x)) dx \quad (16)$$

The right-hand side of the formula above is simply $E[\min\{X_1, X_2\}]$.

It is easily verified that for the asymmetric uniform distributions in Example 1, we have

$$R^{SPAR}(F_1, F_2) = \frac{\omega_2(3\omega_1 - \omega_2)}{6\omega_1}$$

and thus,

$$R^{FPAR}(F_1, F_2) - R^{SPAR}(F_1, F_2) = \frac{(\omega_1 - \omega_2)^2}{6\omega_1}$$

and this is positive as long as $\omega_1 > \omega_2$.

In the next section, we show that the revenue superiority of the first-price auction over its second-price counterpart is general.

5 Revenue Comparison

In this section we establish our main result.

Theorem 3 *The seller's revenue from a first-price auction with resale is at least as great as that from a second-price auction with resale.*

Before proving Theorem 3, some preliminaries are in order.

Calculus of Variations In what follows, we will make use of a simple technique from the calculus of variations that is used to derive the *Euler equation*. (See, for instance, Section 3 in Kamien and Schwartz, 1981).

Consider the integral

$$\Delta = \int_0^{\bar{p}} \Phi(p, M(p), m(p)) dp$$

where $M : [0, \bar{p}] \rightarrow \mathbb{R}$ and $m(p) = M'(p)$. Suppose $Z(p) : [0, \bar{p}] \rightarrow \mathbb{R}$ is a *perturbation* satisfying $Z(0) = Z(\bar{p}) = 0$ and let $z(p) \equiv Z'(p)$. Define

$$\Delta(\varepsilon) = \int_0^{\bar{p}} \Phi(p, M + \varepsilon Z, m + \varepsilon z) dp$$

to be the value of the integral when M is perturbed by εZ . Differentiating with respect to ε ,

$$\Delta'(\varepsilon) = \int_0^{\bar{p}} [\Phi_M Z + \Phi_m z] dp$$

where $\Phi_M \equiv \partial\Phi/\partial M$ and $\Phi_m \equiv \partial\Phi/\partial m$. Integrating by parts,

$$\int_0^{\bar{p}} \Phi_m z dp = \Phi_m Z \Big|_0^{\bar{p}} - \int_0^{\bar{p}} \frac{d}{dp} (\Phi_m) Z dp = - \int_0^{\bar{p}} \frac{d}{dp} (\Phi_m) Z dp$$

since $Z(0) = Z(\bar{p}) = 0$.

Thus we obtain,

$$\Delta'(0) = \int_0^{\bar{p}} \left[\Phi_M - \frac{d}{dp} (\Phi_m) \right] Z dp \tag{17}$$

Notation In the proof below it is convenient to reformulate the problem in terms of the *decumulative* distribution functions. Let

$$H_1(p) \equiv 1 - F_1(p), \quad H_2(p) \equiv 1 - F_2(p) \quad \text{and} \quad H(p) \equiv 1 - F(p)$$

and

$$h_1(p) = H_1'(p), \quad h_2(p) = H_2'(p) \quad \text{and} \quad h(p) = H'(p)$$

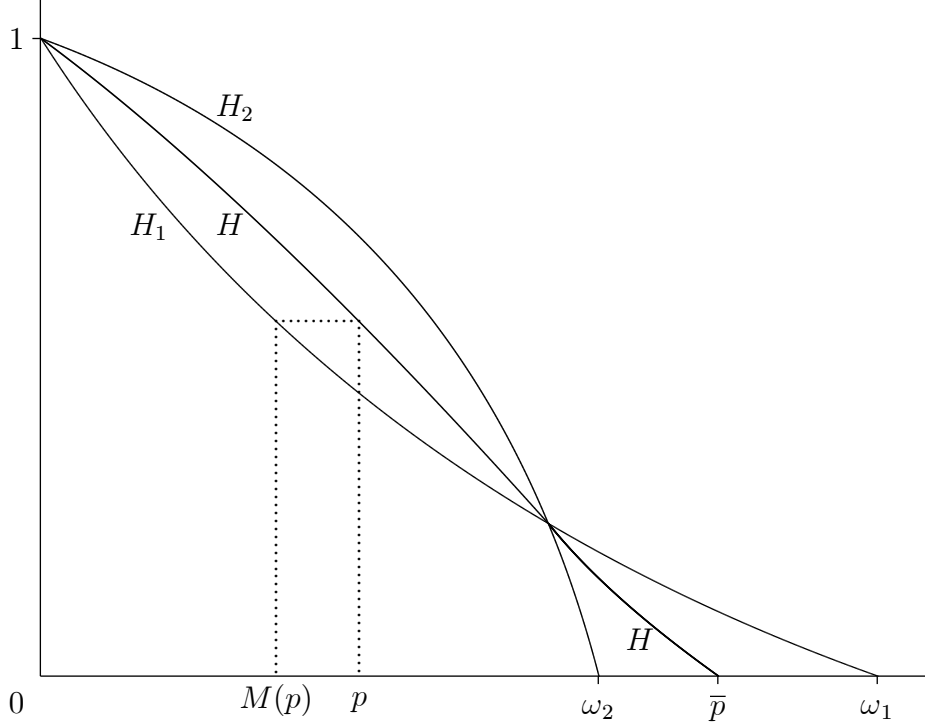


Figure 3: The Function M

Notice that in terms of decumulative functions, (10) can be rewritten as: if $H_i(p) \geq H_j(p)$, then

$$H(p) = H_j \left(p - \frac{H(p) - H_i(p)}{h_i(p)} \right)$$

or equivalently,

$$H(p) = H_i(p) + [p - H_j^{-1}H(p)] h_i(p) \quad (18)$$

The regularity assumption; that is, the monotonicity of virtual values $p + (H_i(p) / h_i(p))$, is equivalent to

$$2h_i(p)^2 - H_i(p) h_i'(p) > 0 \quad (19)$$

Proof of Theorem 3. In terms of the decumulative functions, the difference between the revenue from a FPAR (15) and the revenue from a SPAR (16) is

$$\Delta = \int_0^{\bar{p}} H(p)^2 dp - \int_0^{\omega_2} H_i(x) H_j(x) dx$$

Define the function $M : [0, \bar{p}] \rightarrow [0, \omega_2]$ as follows: if $H_i(p) \geq H_j(p)$, then

$$M(p) = H_j^{-1}(H(p))$$

M is an increasing function satisfying $M(0) = 0$, $M(\bar{p}) = \omega_2$ and since for all p , $H_i(p) \geq H(p) \geq H_j(p)$ it is the case that $M(p) \leq p$. (See Figure 3.) Define $m(p) = M'(p)$ wherever this is well defined. By changing the variable of integration from $x \in [0, \omega_2]$ to $p = M^{-1}(x) \in [0, \bar{p}]$ in the second integral in the expression for Δ , we obtain

$$\begin{aligned}\Delta &= \int_0^{\bar{p}} H(p)^2 dp - \int_0^{\bar{p}} H_i(M(p)) H_j(M(p)) m(p) dp \\ &= \int_0^{\bar{p}} [H(p)^2 - H_i(M(p)) H(p) m(p)] dp\end{aligned}$$

where $H(p)$ is given by

$$H(p) = H_i(p) + [p - M(p)] h_i(p) \quad (20)$$

which, in turn, is directly obtained from (18).

Lemma 1 implies that any H_i and H_j uniquely determine an H satisfying (18) and this in turn uniquely determines an M . Conversely, given an H_i and an M , (20) determines a unique H and this in turn determines a unique H_j since $H_j(x) = H(M^{-1}(x))$.

Note that $H_i = H_j$ if and only if $M(p) = p$ for all p , and in that case, $\Delta = 0$. We will argue that, in general; that is, for all H_i and H_j , $\Delta \geq 0$.

Consider the integrand in the expression for Δ , that is, the function⁷

$$\Phi(p, M, m) = H^2 - H_i(M) H m$$

and the perturbation $Z(p) \equiv p - M(p) \geq 0$. Then define

$$\Delta(\varepsilon) = \int_0^{\bar{p}} \Phi(p, M + \varepsilon Z, m + \varepsilon z) dp$$

to be the revenue difference when M is perturbed slightly in the direction of p . (Figure 4 shows how the perturbation of M in the direction of p moves both H_j and H closer to H_i ; that is, in the direction of increased symmetry.)

We now use (17) to evaluate $\Delta'(0)$. For this we need

$$\begin{aligned}\Phi_M &= 2H \frac{\partial H}{\partial M} - h_i(M) H m - H_i(M) \frac{\partial H}{\partial M} m \\ &= -2H h_i - h_i(M) H m + H_i(M) h_i m\end{aligned} \quad (21)$$

using $\partial H / \partial M = -h_i$, derived from (20).

And since,

$$\Phi_m = -H_i(M) H$$

⁷To ease the notational burden, we suppress the argument p for the remainder of the proof.

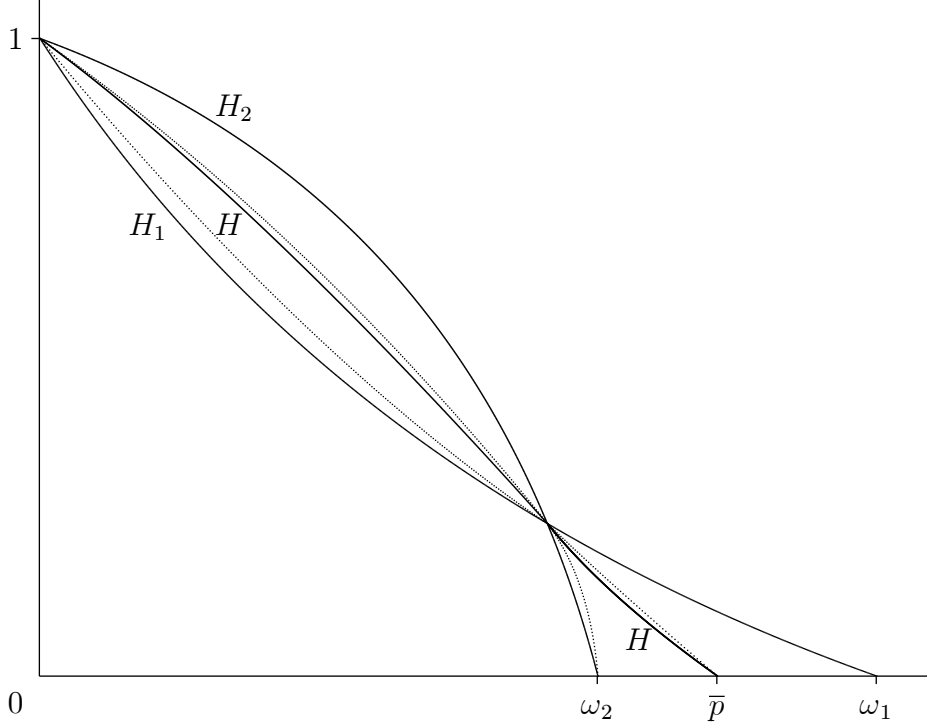


Figure 4: Perturbation of M

we have

$$\frac{d}{dp} (\Phi_m) = -h_i(M) H m - H_i(M) h \quad (22)$$

Now (21) and (22) result in

$$\begin{aligned} \Phi_M - \frac{d}{dp} (\Phi_m) &= -2Hh_i + H_i(M) h_i m + H_i(M) h \\ &= -2Hh_i + H_i(M) h_i m + H_i(M) [2h_i + [p - M] h'_i - h_i m] \\ &= 2 [H_i(M) - H] h_i + H_i(M) [p - M] h'_i \\ &= H_i(M) \left(2 \left[1 - \frac{H}{H_i(M)} \right] h_i - \left[\frac{H_i - H}{h_i} \right] h'_i \right) \end{aligned}$$

where the second equality is obtained by substituting $h = 2h_i + [p - M] h'_i - h_i m$, which is derived from (20). The fourth equality is obtained by substituting $[p - M] = -(H_i - H) / h_i$, again from (20).

But since H_i is decreasing and $M(p) \leq p$, we have $H_i(M(p)) \geq H_i(p)$ and using

the fact that $h_i < 0$,

$$\begin{aligned}\Phi_M - \frac{d}{dp}(\Phi_m) &\leq H_i(M) \left(2 \left[1 - \frac{H}{H_i} \right] h_i - \left[\frac{H - H_i}{h_i} \right] h'_i \right) \\ &= H_i(M) \left[\frac{H_i - H}{H_i h_i} \right] (2h_i^2 - H_i h'_i)\end{aligned}$$

Since F_i is regular, $2h_i^2 - H_i h'_i > 0$, as in (19). Together with $H_i > H$ and $h_i < 0$, this implies

$$\Phi_M - \frac{d}{dp}(\Phi_m) < 0$$

whenever $M(p) < p$.

If $M(p) < p$ for all p , then $Z(p) = p - M(p) > 0$, and so (17) now implies

$$\Delta'(0) = \int_0^{\bar{p}} \left[\Phi_M - \frac{d}{dp}(\Phi_m) \right] [p - M] dp < 0$$

We have shown that given any M such that $M(p) < p$ for all p , a small perturbation $(1 - \varepsilon)M(p) + \varepsilon p$ in the direction of p always decreases Δ . Thus, Δ is minimized at $M = p$. But when $M(p) = p$ for all p , $H_i = H_j$ and this means that $F_1 = F_2$, and in that case, the revenue equivalence principle implies that $\Delta = 0$. We have thus shown that for all F_1 and F_2 that are regular, $\Delta \geq 0$.

This completes the proof of Theorem 3. ■

Remark 2 The proof of Theorem 3 actually shows not only that the difference in revenues between the two auctions is nonnegative but moreover that it is, in fact, increasing in the degree of asymmetry. Suppose that $F_1 \leq F_2$ so that $i = 1$ and $j = 2$ holds everywhere. For fixed F_1 , if F_2 is perturbed in the direction of F_1 , thereby bringing the situation closer to symmetry, then the difference in revenues decreases.

Remark 3 Can Theorem 3 be strengthened so that its conclusion holds for all equilibria (not necessarily ex post) of the second-price auction? The answer is no. With resale, there exist inefficient equilibria of the second-price auction which are not ex post, but result in a higher expected revenue than the first-price auction. Appendix C contains an example.

6 Other Resale Mechanisms

In our analysis of asymmetric auctions with resale, we assumed that post-auction trade took place via a take-it-or-leave-it offer from the winner of the auction. Since all bargaining power resides in the hands of the seller, it is natural to refer to this as the *monopoly* mechanism.

In this section we show that our results are quite robust. We first show that they continue to hold if post-auction trade takes place via a take-it-or-leave-it offer from the loser of the auction. Now all bargaining power lies with the buyer and so by analogy, we will refer to this as the *monopsony* mechanism.

We then extend our results to a class of mechanisms in which the bargaining power is shared, perhaps unequally. Specifically, we consider a mechanism in which with probability k , the seller makes a take-it-or-leave-it offer and with probability $1 - k$, the buyer makes a take-it-or-leave-it offer. We refer to this as the (probabilistic) *k-double auction*.⁸

Recall that in our basic model, at the end of the auction, only the winning bid was announced. When resale takes place via the monopoly mechanism, this is the same as if no bid were announced. This is because information about the winner's bid is irrelevant to the other bidder—he faces a take-it-or-leave-it offer from the winner. In other resale mechanisms, say when the buyer makes a take-it-or-leave-it offer, information regarding the winner's value is no longer irrelevant—for instance, if the winning bid revealed the winner's value, then the loser could extract all surplus from the winner during resale. It can be shown that if the winning bid is announced when resale takes place via monopsony, then there is no monotonic equilibrium.

In the extensions of the basic model that follow, we assume that *no bids are announced* at the end of the auction. Thus only the identity of the winner is commonly known.

6.1 Monopsony Resale

In this case, the loser of the auction can make an offer to buy the object from the winner.

Suppose that the two bidders follow *continuous* and *strictly increasing* inverse bidding strategies ϕ_1 and ϕ_2 , respectively.

Suppose bidder i with value x_i bids b and loses the auction. He infers that bidder j 's value $X_j \geq \phi_j(b)$. If $x_i > \phi_j(b)$, there are potential gains from trade. Let $r_i(b, x_i)$ be the optimal (monopsony) price set by bidder i with value x_i when he bids b and loses. This results in a resale profits of

$$S_i(b, x_i) = \max_r [F_j(r) - F_j(\phi_j(b))] (x_i - r) \quad (23)$$

and the optimal price $r_i(b, x_i)$ must satisfy the first-order condition

$$r - \frac{F_j(\phi_j(b)) - F_j(r)}{f_j(r)} = x_i \quad (24)$$

⁸The term *k-double auction* usually refers to a situation in which the price is a weighted average of the price demanded by the seller and that offered by the buyer. This mechanism is known to have a continuum of equilibria and so is not useful for our analysis.

Regularity again implies that the left-hand side is increasing and so there is a unique solution.

Notice that the monopsony pricing formula in (24) is the *same* as the monopoly pricing formula in (1). The only difference is that now the optimal price $r_i(b, x_i) < x_i$. Notice also that the envelope theorem implies

$$\frac{\partial}{\partial b} S_i(b, x_i) = -f_j(\phi_j(b)) \phi_j'(b) (x_i - r_i(b, x_i))$$

Now fix a b satisfying $\phi_j(b) < \phi_i(b)$. Suppose bidder j follows ϕ_j . Bidder i 's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\Pi_i(c, x_i) = F_j(\phi_j(c)) (x_i - c) + S_i(c, x_i)$$

where $S_i(c, x_i)$ as defined above, is his expected payoff from resale if he loses the auction. Maximizing Π_i , using the envelope theorem to evaluate the derivative of $S_i(c, x_i)$ and writing $r(b) \equiv r_i(b, \phi_i(b))$ leads to

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{r(b) - b}$$

Now consider bidder j . A bid of c satisfying $\phi_j(c) < \phi_i(c)$ when his value is $x_j = \phi_j(b)$ results in an expected payoff of

$$\Pi_j(c, x_j) = F_i(z_i) x_j + \int_{z_i}^{\phi_i(c)} r(\beta_i(x_i)) f_i(x_i) dx_i - F_i(\phi_i(c)) c$$

where z_i is highest type of bidder i whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Since it is optimal for j to bid b , the first-order condition for maximizing Π_j results in the same differential equation as above.

Thus, just as (6), (7) and (8) characterize the equilibrium when resale is via the monopoly mechanism, the following characterize the equilibrium bidding when resale is via the monopsony mechanism.

$$\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{r(b) - b} \quad (25)$$

$$\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{r(b) - b} \quad (26)$$

where if $\phi_j(b) \leq \phi_i(b)$,

$$r(b) = \arg \max_r [F_j(r) - F_j(\phi_j(b))] (\phi_i(b) - r) \quad (27)$$

It should be noted that although the formulae defining the two are similar, in general, $r(b) \neq p(b)$. This implies of course that the equilibrium bidding strategies when

resale is via the monopsony mechanism—that is, the solutions to (25) and (26)—are different from the equilibrium bidding strategies when resale is via the monopoly mechanism. So that there is no ambiguity, let us denote by $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ the equilibrium inverse bidding strategies in a first-price auction in which resale is via monopsony.

The remainder of the analysis parallels that in the case of monopoly *exactly* once the monopoly pricing function $p(b)$ is replaced by the monopsony pricing function $r(b)$. Specifically,

1. As in Theorem 1, the differential equations (25) and (26), together with (27), are both necessary and sufficient for $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ to be an equilibrium.
2. As in Lemma 1, F_1 and F_2 uniquely determine a distribution G of monopsony resale prices where if $F_i(r) \leq F_j(r)$, then

$$G(r) = F_i \left(r - \frac{G(r) - F_j(r)}{f_j(r)} \right) \quad (28)$$

The distribution G has a geometric interpretation similar to that of F in Figure 2. In general, $G \neq F$.

3. As in Theorem 1, there exists an equilibrium with strictly increasing bidding strategies $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ and it is unique in the class of all nondecreasing equilibria.
4. As in Theorem 3, the revenue from a first-price auction with monopsony resale is at least as large as that from a second-price auction with resale.

6.2 Probabilistic k -Double Auctions

In this mechanism, resale takes place as follows. With probability k , the winner of the auction makes a take-it-or-leave-it offer to the loser and with probability $1 - k$ the loser makes a take-it-or-leave-it offer to the winner. Either side may decide not to participate, in which case no transaction takes place.

When $k = 1$, this reduces to the monopoly resale mechanism considered earlier. When $k = 0$, it reduces to the monopsony mechanism of the previous subsection.

Again, fix a b satisfying $\phi_j(b) < \phi_i(b)$. Suppose bidder j follows ϕ_j . Bidder i 's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\begin{aligned} \Pi_i(c, x_i) &= (x_i - c) F_j(\phi_j(c)) \\ &\quad + k \int_{\phi_j(c)}^{\omega_j} \max \{ x_i - p(\beta_j(x_j)), 0 \} f_j(x_j) dx_j + (1 - k) S_i(c, x_i) \end{aligned}$$

This is because if i loses, with probability k bidder j will offer to sell him the object at price $p(\beta_j(x_j))$. With probability $1 - k$, he will offer to buy from the other bidder

and obtain a profit of $S_i(c, x_i)$, as defined in (23). Differentiating with respect to c and using the equilibrium condition $x_i = \phi_i(b)$ results in

$$(kp(b) + (1 - k)r(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) = 0$$

If we write $s(b) = kp(b) + (1 - k)r(b)$, then

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{s(b) - b}$$

Now consider bidder j . A bid of c satisfying $\phi_j(c) < \phi_i(c)$ when his value is $x_j = \phi_j(b)$ results in an expected payoff of

$$\Pi_j(c, x_j) = kR_j(c, x_j) + (1 - k) \int_{z_i}^{\phi_i(c)} r(\beta_i(x_i)) f_i(x_i) dx_i - F_i(\phi_i(c)) c$$

where z_i is lowest type of bidder i whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Differentiating, and using the equilibrium condition $x_j = \phi_j(b)$, leads to

$$[kp(b) + (1 - k)r(b)] f_i(\phi_i(b)) \phi_i'(b) - b f_i(\phi_i(b)) \phi_i'(b) - F_i(\phi_i(b)) = 0$$

which again results in

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{s(b) - b}$$

Thus the equilibrium when resale is via a k -double auction is characterized by

$$\begin{aligned} \frac{d}{db} \ln F_1(\phi_1(b)) &= \frac{1}{s(b) - b} \\ \frac{d}{db} \ln F_2(\phi_2(b)) &= \frac{1}{s(b) - b} \end{aligned}$$

where $s(b) = kp(b) + (1 - k)r(b)$ and $p(b)$ and $r(b)$ are given in (8) and (27), respectively.

Once again, in general, $p(b) \neq r(b)$ and so for $k \in (0, 1)$, $s(b)$ is distinct from both $p(b)$ and $r(b)$. Thus the equilibrium bidding strategies are now different from both those in the case of monopoly resale and those in the case of monopsony resale. This in turn implies that the pricing functions $p(b)$ and $r(b)$ in the case of a k -double auction are also different from those resulting in the case of a pure monopoly or a pure monopsony.

Define $\widehat{\varphi}_1, \widehat{\varphi}_2$ to be the equilibrium inverse bidding strategies in a first-price auction in which resale is via the k -double auction. Similarly, define $\widehat{p}(b)$ and $\widehat{r}(b)$ to be the monopoly and monopsony pricing functions in the k -double auction. Thus $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{p}, \widehat{r}$ are the simultaneous solutions to the two differential equations above together with (8) and (27). For notational consistency, let $\widehat{s}(b) = k\widehat{p}(b) + (1 - k)\widehat{r}(b)$.

6.2.1 Distribution of $\widehat{s}(b)$

We wish to determine the distribution of the random variable $\widehat{s}(b)$. As before, the differential equations imply once again that the distributions of bids of the two bidders are identical; that is, $F_1(\widehat{\varphi}_1(b)) = F_2(\widehat{\varphi}_2(b))$. Let $A(b) \equiv F_j(\widehat{\varphi}_j(b))$ be the common distribution of bids.

Assume, without loss of generality, that $\widehat{\varphi}_j(b) < \widehat{\varphi}_i(b)$ and notice that $\widehat{p}(b), \widehat{r}(b)$ and $\widehat{s}(b)$ satisfy:

$$\widehat{\varphi}_j(b) = \widehat{p}(b) - \frac{F_i(\widehat{\varphi}_i(b)) - F_i(\widehat{p}(b))}{f_i(\widehat{p}(b))} \quad (29)$$

$$\widehat{\varphi}_i(b) = \widehat{r}(b) - \frac{F_j(\widehat{\varphi}_j(b)) - F_j(\widehat{r}(b))}{f_j(\widehat{r}(b))} \quad (30)$$

$$\widehat{s}(b) = k\widehat{p}(b) + (1-k)\widehat{r}(b) \quad (31)$$

Define distributions \widehat{F}, \widehat{G} and L as follows

$$\widehat{F}(\widehat{p}(b)) = \widehat{G}(\widehat{r}(b)) = L(\widehat{s}(b)) = A(b) \quad (32)$$

F, G and L are well defined since $\widehat{p}(\cdot), \widehat{r}(\cdot)$ and $\widehat{s}(\cdot)$ are increasing functions.

Note that (29) and (30) then can be rewritten as follows:

$$\widehat{F}(p) = F_j \left(p - \frac{\widehat{F}(p) - F_i(p)}{f_i(p)} \right) \quad (33)$$

$$\widehat{G}(r) = F_i \left(r - \frac{\widehat{G}(r) - F_j(r)}{f_j(r)} \right) \quad (34)$$

But now notice that since (33) is the same as (10) and F was uniquely determined there, $\widehat{F} = F$, the distribution of prices when resale is via monopoly. Similarly, (34) is the same as (28) and so $\widehat{G} = G$, the distribution of prices when resale is via monopsony. We thus obtain the conclusion that even though, in general, $\widehat{p} \neq p$ and $\widehat{\varphi}_i \neq \phi_i$, we have

$$F(\widehat{p}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } F(p(b)) = F_i(\phi_i(b))$$

Similarly, even though, in general, $\widehat{r} \neq r$ and $\widehat{\varphi}_i \neq \widetilde{\varphi}_i$,

$$G(\widehat{r}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } G(r(b)) = F_i(\widetilde{\varphi}_i(b))$$

Finally, since $F = \widehat{F}$ and $G = \widehat{G}$, (32) implies that for all b ,

$$L^{-1}(A(b)) = kF^{-1}(A(b)) + (1-k)G^{-1}(A(b))$$

Since $A(b)$ varies from 0 to 1, we obtain

$$L^{-1}(q) = kF^{-1}(q) + (1-k)G^{-1}(q) \quad (35)$$

for all $q \in [0, 1]$, where F and G satisfy (33) and (34), respectively.

6.2.2 Revenue from FPA with Resale via k -Double Auction

Lemma 2

$$\int_0^{\bar{s}} (1 - L(s))^2 ds = k \int_0^{\bar{p}} (1 - F(p))^2 dp + (1 - k) \int_0^{\bar{r}} (1 - G(r))^2 dr$$

Proof. Let $L(s) = q$, then from (35) we obtain

$$s = L^{-1}(q) = kF^{-1}(q) + (1 - k)G^{-1}(q)$$

so that

$$ds = \left(\frac{k}{f(F^{-1}(q))} + \frac{1 - k}{g(G^{-1}(q))} \right) dq$$

By changing the variable of integration from $s \in [0, \bar{s}]$ to $q = L(s) \in [0, 1]$ we obtain

$$\begin{aligned} \int_0^{\bar{s}} (1 - L(s))^2 ds &= \int_0^1 (1 - q)^2 \left(\frac{k}{f(F^{-1}(q))} + \frac{1 - k}{g(G^{-1}(q))} \right) dq \\ &= k \int_0^1 \frac{(1 - q)^2}{f(F^{-1}(q))} dq + (1 - k) \int_0^1 \frac{(1 - q)^2}{g(G^{-1}(q))} dq \end{aligned}$$

Changing the variables again from $q \in [0, 1]$ to $p = F^{-1}(q) \in [0, \bar{p}]$ in the first integral and $r = G^{-1}(q) \in [0, \bar{r}]$ in the second, we obtain the required equality. ■

The expected revenue of the original seller when resale is via the k -double auction is given by

$$\int_0^{\bar{s}} (1 - L(s))^2 ds$$

where L satisfies (35).

We have already shown in Theorem 3 that

$$\int_0^{\bar{p}} (1 - F(p))^2 dp \geq \int_0^{\omega_2} (1 - F_1(x))(1 - F_2(x)) dx$$

and it is similarly the case that

$$\int_0^{\bar{r}} (1 - G(r))^2 dr \geq \int_0^{\omega_2} (1 - F_1(x))(1 - F_2(x)) dx$$

Lemma 2 now implies that the revenue from a first-price auction with resale via the k -double auction is also greater than or equal to the revenue from a second-price auction.

The following theorem extends our main result.

Theorem 4 *The seller's revenue from a first-price auction with resale via a probabilistic k -double auction is at least as great as that from a second-price auction.*

Theorem 4 subsumes the case of monopoly (when $k = 1$) and the case of monopsony (when $k = 0$). The reader may thus wonder why we did not establish only the more general result. The proof of Theorem 4, however, makes use of both Theorem 3 and its counterpart for the monopsony mechanism.

Finally, one may well ask whether the revenue ranking is valid for arbitrary (balanced-budget) resale mechanisms. In other words, suppose that the resale mechanism is of the direct form (Q, T) , where $Q(z_b, z_s)$ is the probability that the object is transferred from the seller to the buyer when (z_b, z_s) are the values reported by the two parties and $T(z_b, z_s)$ is the payment from the buyer to the seller. As usual, we ask that the mechanism be incentive compatible and individually rational. Does Theorem 4 extend to this general specification? Clearly, the answer is no. For instance, the result does not hold for the “no-trade” mechanism ($Q = 0$ and $T = 0$) that, in effect, bans resale—as Vickrey’s (1961) example shows. This suggests that the revenue ranking to hold, the probability of a resale transaction should not be “too low.”

7 Conclusion

We have shown that a consideration of resale possibilities allows for a simpler characterization of equilibrium strategies in first-price auctions than available when resale is not admitted. In our model, equilibrium strategies can be explicitly computed in a relatively simple manner as in the proof of Theorem 1. Moreover, we obtain a general revenue ranking result between first- and second-price auctions that is not available in the standard model. Thus this appears to be one of those happy circumstances where complicating the model with a real-world feature—resale—actually simplifies the analysis.

In this paper, we have restricted attention to the case of two bidders. Considering resale when there are three or more bidders poses some conceptual difficulties.⁹ There are simply too many modelling choices and the challenge is to model resale in a way that is both realistic and analytically tractable. To fix ideas, consider a first-price auction with three bidders and suppose that bidder 3 wins the auction. At the resale stage, among the many available options are:

1. Bidder 3 invites both bidders 1 and 2 to participate in an auction.
2. Bidder 3 approaches one of the bidders, say bidder 1, and makes a take-it-or-leave-it offer.

The results of this paper have a direct bearing on any analysis of the first option. This is because the resale auction involves only two bidders. A model with these

⁹Zheng (2002) also finds that when there are three or more bidders, Myerson’s optimal auction is robust to resale only under stringent conditions (see Mylovanov and Tröger (2005)).

features could, however, be legitimately criticized on the grounds that most resale transactions are bilateral—we simply do not observe multiple rounds of auctions with the winner in each round holding an auction for the remaining bidders. But it is not clear why the bilateral transaction should consist of only one take-it-or-leave-it offer. Bidder 3 could approach bidder 1 and if his offer is refused, then approach bidder 2. And this raises the question of the order in which bidders 1 and 2 should be approached. We hope to consider some of these issues in future work.

A Appendix: Proofs from Section 3

This appendix contains proofs of results on first-price auctions with resale.

A.1 Proof of Proposition 1

Proof. The necessity of (6) and (7) for all $b \in [0, \bar{b}]$ has already been shown. It remains to show that these are sufficient.

Suppose bidder j follows the equilibrium inverse bidding strategy ϕ_j . We will argue that when bidder i has a value of x_i , he cannot do better than to bid b such that $\phi_i(b) = x_i$. We do this by showing that neither underbidding nor overbidding can be profitable.

Notice that the differential equations can be rewritten as: for $j = 1, 2$

$$(p(b) - b)f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) = 0 \quad (36)$$

CASE A (UNDERBIDDING): Suppose bidder i bids b such that $\phi_i(b) < x_i$.

CASE A1: $\phi_j(b) < \phi_i(b) < x_i$. If i wins the auction with a bid of b , then his payoff is simply $(x_i - b)$ since there are no benefits to reselling. If i loses, however, j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so i 's payoff is $\max\{x_i - p(\beta_j(x_j)), 0\}$. Thus i 's expected payoff is

$$\Pi_i(b, x_i) = (x_i - b)F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j) dx_j$$

Differentiating with respect to b and using (36), results in

$$\frac{\partial \Pi_i}{\partial b} = (p(b) - b)f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) = 0$$

CASE A2: $\phi_i(b) \leq \phi_j(b) < x_i$. If i wins the auction with a bid of b , then his payoff is simply $(x_i - b)$ since again there are no benefits to reselling. Similarly, if i loses, bidder j will not offer to sell the object to him since from j 's perspective, there appear to be no benefits to selling to i . Thus i 's expected payoff is simply

$$\Pi_i(b, x_i) = (x_i - b)F_j(\phi_j(b))$$

and so again by using (36),

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial b} &= (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&\geq (\phi_j(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&= 0
\end{aligned}$$

CASE A3: $\phi_i(b) < x_i \leq \phi_j(b)$. If i wins the auction with a bid of b , then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is

$$R_i(b, x_i) = \max[F_j(\phi_j(b)) - F_j(p)]p + F_j(p)x_i$$

If i loses, bidder j will not offer to sell the object to him since from j 's perspective, there appear to be no gains from trade. Thus i 's expected payoff from bidding b is

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and using the envelope theorem as in (4) and the fact that $p_i(b, x_i) \geq p_i(b, \phi_i(b)) \equiv p(b)$,

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial b} &= (p_i(b, x_i) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&= 0
\end{aligned}$$

CASE B (OVERBIDDING): Suppose bidder i bids b such that $x_i < \phi_i(b)$.

CASE B1: $\phi_j(b) < x_i < \phi_i(b)$. If i wins the auction with a bid of b , then his payoff is simply $(x_i - b)$ since there is no benefit from reselling to j . On the other hand, if i loses, j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so i 's payoff if he loses is $\max\{x_i - p(\beta_j(x_j)), 0\}$. Thus i 's expected payoff from bidding b is

$$\Pi_i(b, x_i) = (x_i - b) F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j) dx_j$$

Differentiating with respect to b ,

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial b} &= (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) - \max\{x_i - p(b), 0\} f_j(\phi_j(b)) \phi_j'(b) \\
&\leq (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) - (x_i - p(b)) f_j(\phi_j(b)) \phi_j'(b) \\
&= (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\
&= 0
\end{aligned}$$

CASE B2: $x_i \leq \phi_j(b) < \phi_i(b)$. If i wins the auction with a bid of b , then he may resell it to bidder j since again there are potential gains from trade. If he loses, bidder j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ but this price will always exceed x_i and so i will refuse the offer. Thus i 's expected payoff from bidding b is just

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and again using (4) and the fact that $p_i(b, x_i) \leq \phi_j(b) \leq p_j(b, \phi_j(b)) \equiv p(b)$,

$$\begin{aligned} \frac{\partial \Pi_i}{\partial b} &= (p_i(b, x_i) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &\leq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &= 0 \end{aligned}$$

CASE B3: $x_i < \phi_i(b) \leq \phi_j(b)$. If i wins the auction with a bid of b , then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is the monopoly profit $R_i(b, x_i)$. If he loses, bidder j will not offer to sell the object to him since from j 's perspective, there appear to be no gains from trade. Thus i 's expected payoff from bidding b is again

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and the argument is the same as in Case B2, except that now $p_i(b, x_i) \leq p_i(b, \phi_i(b)) \equiv p(b)$.

We have thus argued that for all b such that $\phi_i(b) < x_i$, $\frac{\partial \Pi_i}{\partial b} \geq 0$ and for all b such that $\phi_i(b) > x_i$, $\frac{\partial \Pi_i}{\partial b} \leq 0$. Thus bidding a b such that $\phi_i(b) = x_i$ is a best response to ϕ_j . ■

A.2 Proof of Lemma 1

Proof. Fix a p and note that

$$\Psi(p, q) \equiv F_j \left(p - \frac{q - F_i(p)}{f_i(p)} \right)$$

is a strictly decreasing function of q , where $F_i(p) \leq q \leq 1$.

If $F_i(p) = F_j(p)$, then $F(p) = F_i(p)$ also. If $F_i(p) < F_j(p)$, then $\Psi(p, F_i(p)) > F_i(p)$ and $\Psi(p, F_j(p)) < F_j(p)$. Thus for every p , there exists a unique fixed-point $q \in (F_i(p), F_j(p))$ such that $\Psi(p, q) = q$ and by (10) $F(p) \equiv q$. Thus, $F_i(p) \leq F(p) \leq F_j(p)$ and the inequalities are strict if $F_i(p) < F_j(p)$.

Clearly, $F(0) = 0$. We now argue that F is strictly increasing. It suffices to show that $F(p') < F(p'')$ for $p' < p''$ such that for all $p \in [p', p'']$, $F_i(p) \leq F_j(p)$. The regularity of F_i (see Section 2) implies that for all q ,

$$p'' - \frac{q - F_i(p'')}{f_i(p'')} > p' - \frac{q - F_i(p')}{f_i(p')}$$

Thus, if $p' < p''$, then for all q , $\Psi(p'', q) > \Psi(p', q)$. This implies that if $\Psi(p', q') = q'$ and $\Psi(p'', q'') = q''$, then $q'' > q'$. So $F(p'') > F(p')$. Finally, note that since $\omega_1 \geq \omega_2$, F has support $[0, \bar{p}]$, where \bar{p} satisfies

$$\omega_2 = \bar{p} - \frac{1 - F_1(\bar{p})}{f_1(\bar{p})}$$

We have thus shown that F is a well-defined distribution function over $[0, \bar{p}]$. ■

A.3 Proof of Theorem 2

It has already been established that the equilibrium constructed in Theorem 1 is unique in the class of equilibria with strictly increasing strategies. Here we complete the proof Theorem 2 by showing that every equilibrium in nondecreasing strategies must, in fact, have strictly increasing bidding strategies.

Lemma 3 *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are continuous.*

Proof. Suppose that there exists an $x_i > 0$ such that $\lim_{x \uparrow x_i} \beta_i(x) = b' < b'' = \lim_{x \downarrow x_i} \beta_i(x)$. First, note that in that case bidder j also does not bid between b' and b'' ; that is, there does not exist an x_j such that $b' < \beta_j(x_j) < b''$. Otherwise, bidder j with value x_j could increase his payoff by decreasing his bid to $\beta_j(x_j) - \varepsilon > b'$. This change does not affect his payoff if he were to lose but increases it by ε if he were to win (which happens with positive probability since $x_i > 0$). Second, note that bidder j bids b' or lower with positive probability. Otherwise, $\lim_{x \rightarrow 0} \beta_j(x) \geq b''$ and bidder j with value x_j close to zero can improve his payoff by reducing his bid to b' . This is because bidder j would gain at least $b'' - b'$ whenever bidder i 's value was between 0 and x_i and suffer only a small loss in the cases when bidder i 's value is just slightly above x_i . So there does not exist an x_j such that $b' < \beta_j(x_j) < b''$.

Now consider bidder i with a value slightly greater than x_i , say $x_i + \delta$. By reducing his bid from $\beta_i(x_i + \delta) \geq b''$ to b' , bidder i could increase his payoff. Once again, this change does not affect his payoff if he were to lose and increases it by at least $b'' - b'$ if he were to win (which happens with positive probability). Thus for δ small enough, bidder i with value $x_i + \delta$ has a profitable deviation.

We have argued that the bidding strategies β_i are continuous at any $x_i > 0$. It remains to argue that they are also continuous at 0.

Suppose that $\lim_{x \rightarrow 0} \beta_i(x) = b_0 > 0$ and without loss of generality, suppose that for some small δ , it is the case that for all $x \in (0, \delta)$, $\beta_i(x) \geq \beta_j(x)$. Then we must have that $\lim_{x \rightarrow 0} \beta_j(x) = b_0$ also. Otherwise, bidder i with a value close to zero could reduce his bid and improve his payoff. If β_i is increasing in $(0, \delta)$, then the same argument that shows that $\beta_i(0) = 0$ when both strategies are increasing

shows that this is impossible. If both β_i and β_j are constant over $(0, \delta)$, then bidder i with value close to zero can improve his payoff by increasing his bid slightly. Thus $\lim_{x \rightarrow 0} \beta_i(x) = 0$. ■

Lemma 4 *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are strictly increasing.*

Proof. Suppose that there is an interval $[x', x'']$ such that for all $x_i \in (x', x'')$, $\beta_i(x_i) = b > 0$; that is, bidder i 's strategy is constant. Consider bidder j with value x_j such that $\lim_{x \uparrow x_j} \beta_j(x) = b$. For x close to x_j , bidder j can improve his payoff by bidding higher than b . This is because he then wins whenever $X_i \in (x', x'')$ and the loss is arbitrarily small. If $b = 0$, then bidder j with a small value x_j can improve his payoff by reducing his bid. ■

B Appendix: Proofs from Section 4

B.1 Proof of Proposition 4

The proof that there is a unique ex post equilibrium in the second-price auction with resale follows from the lemmas below.

We suppose that equilibrium bidding strategies are right continuous. Since the expected payoff functions of the bidders are continuous in equilibrium, if there is a discontinuity in the bidding strategies, bidders must be indifferent between bidding the right and left limits. Therefore, focusing on equilibria with right-continuous bidding strategies is without loss of generality.

Lemma 5 *Suppose β_1 and β_2 are ex post equilibrium bidding strategies in a SPAR. For all x_i and for all $b \leq \beta_i(x_i)$, the set $\{x_j : \beta_j(x_j) = b \text{ and } x_j > x_i\}$ is either a singleton or empty.*

Proof. If there are two points, say x'_j and x''_j in the set, then in equilibrium, bidder i wins both when the realization is (x_i, x'_j) and when it is (x_i, x''_j) . In both cases, there are gains from trade but the price $p_i(b_j, x_i)$ he will set will be the same. Thus in at least one of the two cases, i is sure to regret his choice. If his offer is accepted, he regrets setting too low a price. If the offer is rejected, he regrets setting too high a price. ■

A simple but important consequence of Lemma 5 is that if j loses in the auction, his payoff must be zero. Either bidder i makes no offer to him or makes an offer equal to x_j .

Lemma 6 *Suppose β_1 and β_2 are ex post equilibrium bidding strategies in a SPAR. Then $x_i > x_j$ implies $\beta_i(x_i) \geq \beta_j(x_j)$.*

Proof. Suppose not. Then there exist x_i and x_j such that $x_i > x_j$ but $\beta_i(x_i) < \beta_j(x_j)$.

Lemma 5 implies that when the realization is (x_i, x_j) , bidder i 's payoff is zero. Thus, $\beta_j(x_j) \geq x_i$. This is because if $\beta_j(x_j) < x_i$, then bidder i would regret not bidding between $\beta_j(x_j)$ and x_i and thereby winning.

Now we argue that for all x'_i such that $x_j < x'_i < x_i$, $\beta_i(x'_i) < \beta_j(x_j)$. Otherwise, if $\beta_i(x'_i) \geq \beta_j(x_j)$ then $\beta_i(x'_i) \geq \beta_j(x_j) \geq x_i > x'_i$. If for the realization (x'_i, x_j) , bidder i wins then he pays $\beta_j(x_j)$, which is more than his value. Moreover, since $x_j < x'_i$, he cannot gain from resale. If for the realization (x'_i, x_j) , bidder i loses (because the bids are tied and he loses the coin toss), then again he gets a payoff of 0 and a negative expected payoff. Thus bidder i is better off by bidding 0.

Finally, consider the realization (x_i, x_j) . Suppose bidder i deviates to $\beta_i(x'_i)$ for x'_i such that $x_j < x'_i < x_i$. We know that $\beta_i(x'_i) < \beta_j(x_j)$ so that he will lose. But now from Lemma, that bidder j will believe that his value is $x'_i > x_j$ and so he will be offered a price of $x'_i < x_i$. Thus when his value is x_i , bidding $\beta_i(x'_i)$ is better than bidding $\beta_i(x_i)$. ■

Lemma 7 *Suppose β_1 and β_2 are ex post equilibrium bidding strategies in a SPAR. Then for all $x \in [0, \omega_2)$, $\beta_1(x) = x = \beta_2(x)$.*

Proof. First, note that in an ex post equilibrium, bidders must follow the same bidding strategy; that is, for all $x \in [0, \omega_2)$, $\beta_1(x) = \beta_2(x)$. This is because Lemma 6 implies that for all $\varepsilon > 0$, $\beta_i(x + \varepsilon) \geq \beta_j(x)$. By right-continuity, $\beta_i(x) \geq \beta_j(x)$. Interchanging the roles of i and j then results in $\beta_1(x) = \beta_2(x)$.

Now suppose $\beta_1(x) = \beta_2(x) > x$. By right-continuity, for all $\varepsilon > 0$ small enough, $\beta_1(x + \varepsilon) > x + \varepsilon$ and $\beta_1(x) > x + \varepsilon$. Consider a realization $(x, x + \varepsilon)$ and suppose that i wins the auction. Since the price he pays exceeds both values, his overall payoff is negative so he is better off by bidding 0. Hence, $\beta_1(x) = \beta_2(x) \leq x$.

Next, suppose $\beta_1(x) = \beta_2(x) < x$. By right-continuity, for all $\varepsilon > 0$ small enough, $\beta_1(x + \varepsilon) < x + \varepsilon$. Consider a realization $(x, x + \varepsilon)$ and suppose that j loses the auction. From Lemma 5 his payoff is zero. If he bids b_j sufficiently close to but less than x , he wins and obtains a positive payoff. Hence, $\beta_1(x) = \beta_2(x) \geq x$. ■

C Appendix: High Revenue Equilibria of SPAR

In this appendix we present an example of a (perfect Bayesian) equilibrium of the second-price auction with resale which yields higher revenues than the first-price auction. Of course, this equilibrium does not have the ex post property nor is it efficient.

Example 4 *Let $F_1(x) = x$ and $F_2(x) = x^{\frac{1}{3}}$, both defined over $[0, 1]$.*

Consider the bidding strategies: for all $x \in [0, 1]$.

$$\begin{aligned}\beta_1(x) &= \frac{1}{4} \\ \beta_2(x) &= 1\end{aligned}$$

If bidder 2 bids anything other than 1, then bidder 1 believes that 2's value is 0. Similarly, if bidder 1 bids anything other than $\frac{1}{4}$, then bidder 2 believes that 1's value is 0.

This, together with optimal behavior in the resale stage, constitutes a (perfect Bayesian) equilibrium of the second-price auction with resale.

Bidder 2 wins for sure and when his value is x_2 , has an expected payoff of $\left(\frac{1+x_2}{2}\right)^2 - \frac{1}{4} \geq 0$. If bidder were to deviate and lose the auction by bidding below $\frac{1}{4}$, he would not be offered the object at the resale stage and so his payoff would be zero. Thus bidder 2 does not gain by deviating.

Given that bidder 2 bids 1, bidder 1 cannot gain by deviating either.

The revenue of the auctioneer in this equilibrium is clearly $\frac{1}{4}$.

In the FPAR, it may be verified that

$$F(p) = \left(p + \left(p^2 + \frac{1}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - \frac{1}{3} \left(p + \left(p^2 + \frac{1}{27}\right)^{\frac{1}{2}}\right)^{-\frac{1}{3}}$$

and the resulting revenue is 0.1268, which is less than the revenue from the SPAR.

References

- [1] Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–889.
- [2] Blume, A. and P. Heidhues (2004): "All Equilibria of the Vickrey Auction," *Journal of Economic Theory*, 114, 170–177.
- [3] Bulow, J., and J. Roberts (1989): "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97, 1060–1090.
- [4] Cantillon, E. (2005): "The Effects of Bidder Asymmetries on Expected Revenue in Auctions," Working Paper, Harvard Business School.
- [5] Cheng, H. (2005): "Ranking Sealed High-Bid and Open Asymmetric Auctions," IEPR Working Paper 05.26, University of Southern California.
- [6] Garrat, R. and T. Tröger (2006): "Speculation in Standard Auctions with Resale," *Econometrica* (forthcoming).
- [7] Griesmer, J., R. Levitan and M. Shubik (1967): "Toward a Study of Bidding Processes, Part IV—Games with Unknown Costs," *Naval Research Logistics Quarterly*, 14, 415–434.

- [8] Gupta, M. and B. Lebrun (1999): “First Price Auctions with Resale,” *Economics Letters*, 64, 181–185.
- [9] Haile, P. (2003): “Auctions with Private Uncertainty and Resale Opportunities,” *Journal of Economic Theory*, 108, 72–110.
- [10] Jackson, M. and J. Swinkels (2005): “Existence of Equilibrium in Single and Double Private Value Auctions,” *Econometrica*, 73, 93–139.
- [11] Kamien, M. and N. Schwartz (1981): *Dynamic Optimization*, New York: Elsevier.
- [12] Lebrun, B. (2006): “Uniqueness of Equilibrium in First-Price Auctions,” *Games and Economic Behavior* (forthcoming).
- [13] Marshall, R., M. Meurer, J.-F. Richard, and W. Stromquist (1994): “Numerical Analysis of Asymmetric First Price Auctions,” *Games and Economic Behavior*, 7, 193–220.
- [14] Maskin, E. and J. Riley (2000): “Asymmetric Auctions,” *Review of Economic Studies*, 67, No. 3, 413–438.
- [15] Maskin, E. and J. Riley (2003): “Uniqueness of Equilibrium in Sealed High-Bid Auctions,” *Games and Economic Behavior*, 45, 395–409.
- [16] Mylovanov, T. and T. Tröger (2005): “A Note on Zheng’s Conditions for Implementing an Optimal Auction with Resale,” Working Paper, University of Bonn, October.
- [17] Myerson, R. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6, No. 1, 58–73.
- [18] Plum, M. (1992): “Characterization and Computation of Nash Equilibria for Auctions with Incomplete Information,” *International Journal of Game Theory*, 20, 393–418.
- [19] Reny, P. (1999): “On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games,” *Econometrica*, 67, 1029–1056.
- [20] Vickrey, W. (1961): “Counterspeculation, Auctions and Competitive Sealed Tenders,” *Journal of Finance*, 16, 8–37.
- [21] Zheng, C. (2002): “Optimal Auctions with Resale,” *Econometrica*, 70, 2197–2224.