Arbitrage Networks*

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1 Introduction

The Arrow-Debreu-Radner (ADR) model describes a world in which all economic actors are price-takers and in which all claims (possibly incomplete) and commodities are exogenously given and are traded on a centralized exchange, with a Walrasian auctioneer determining one market price vector clearing all markets simultaneously. The simplifications involved in this setup have allowed the model to become a useful benchmark in economics.

Clearly not all actual markets correspond exactly or even approximately to such an idealization, and we would like to argue in this paper that global financial markets should be modeled based upon an extension of the ADR model in at least three directions.

First, while most retail investors in financial markets can be safely considered to be price-takers, agents such as universal banks, investment banks, market makers, mutual and hedge fund managers, insurance companies and the like do exert considerable influence on markets and must be presumed to be strategic rather than price-taking.

Second, not all assets and commodities in the entire world are traded simultaneously on one single giant exchange. Assets are traded on a variety of trading posts, such as stock exchanges, options and futures exchanges, as well as over-the-counter (OTC), i.e. in direct and private arms-length transactions bypassing formal exchanges. A large fraction of trades are OTC, and in this category one can include many derivatives deals, foreign exchange dealings, upstairs trading, block trading, bank loans and deposits, private placements of securities, book building such as in primary and secondary stock issues, and so forth. We refer to such trading posts as *exchanges.* Various clientèles trade on different exchanges, and very few retail clients trade on more than one exchange, let alone on all of them simultaneously.

This market segmentation leads to asset price characteristics distinct from those that the ADR model can generate. The usefulness of such a setup has been recognized long ago, as documented for instance in the success of the market segmentation hypothesis (Culbertson (1957)) and the preferred habitat hypothesis (Modigliani and Sutch (1966)) in fixed income analytics. More generally, the assumption of market segmentation implies that asset prices are determined locally and that as a result overall asset prices need not be contained in the set of no-arbitrage prices. This opens up the possibility for some sophisticated players to profit from said opportunities by intermediating and facilitating trade in order to exploit gains from trade. These large players, which we shall simply refer to as *arbitrageurs* in this paper, have a well-defined objective function even in the presence of exploitable arbitrage opportunities, since the awareness of a market impact naturally bounds their trades, for else arbitrage opportunities vanish and no profit at all can be reaped.

The fact that markets are decentralized across various exchanges leaves open the question as to how the global arbitrageurs link exchanges and investors. In Rahi and Zigrand (2005), all arbitrageurs are active on all exchanges. We refer to this scenario as "universal arbitrage." In actual trading networks, however, even large traders only operate on a few exchanges at best. In the current paper, we shall from the outset allow arbitrageurs to only link two exchanges, but let them choose which ones. As a result, the active links of the network and the number of traders on each such link emerge endogenously at a Nash equilibrium of the network formation game.

Third, while many of traded securities are issued by firms and other non-financial institutions, a considerable number of securities are in fact issued by what we call arbitrageurs. One can name assets issued by members of futures and options exchanges (who profit from spreads both within and across exchanges, say by arbitraging futures with their underlying constituents), by banks and related financial institutions (for instance issuance of CDOs and all OTC derivatives; even the advice by investment banks as to the optimal capital structure of firms opens up the possibility of capital structure arbitrage). Which assets are innovated and marketed to which clientèle group is an important consideration when trying to understand the global financial structure. How the various markets are tied in with each other is endogenous and reflects credit, strategic, replicating portfolio and other considerations. In our model, beyond endogenizing the network of trades, we allow arbitrageurs to design their own securities on each of the exchanges on which they are active. The security structure is determined at a Nash equilibrium of the security design game. Given an equilibrium asset structure, arbitrageurs' trades then constitute a Nash equilibrium of the trading game.

An overall equilibrium is therefore a subgame perfect outcome of a three-stage game. The backwards order by which we solve the game is as follows. First investors solve for their portfolio demands given the asset structure and given the supplies of assets by arbitrageurs, and arbitrageur trades are determined at a Nash equilibrium of the trading game, taking as given the demand function of investors, the available assets, and the network structure. In the next stage arbitrageurs determine the equilibrium asset structure in the security design game given the network, and finally the network (the distribution of arbitrageurs across all permissible links) is determined at a Nash equilibrium of the network formation game.

The questions we would like to ask are the following. What is the equilibrium network, and how do the equilibrium asset structure and asset trades depend on the network? How integrated can we expect the global economy to be? Can different exchanges be integrated to a different extent? To which equilibrium does the economy converge as the number of arbitrageurs grows without limit? When will the equilibrium in the limit be integrated and when will the global economy merely be a collection of disconnected subnetworks? How is the network related to the extent of gains from trade between trading locations and their depth (as measured by the price impact of an additional unit of trade). How is the network affected by externalities exerted by arbitrageurs active on different links? What kind of network architecture (i.e. the set of links that are permissible) aligns the interests of arbitrageurs thereby promoting efficiency?

We are not aware of any papers that have studied these questions in the context of asset markets. There is a large literature on networks in other settings. For example, in Bala and Goyal (2000) and Goyal and Joshi (2005), agents form links with other agents in an abstract game. Incentives to form links depend solely on the number of links the player as well as the potential partner has. Here, in contrast, it crucially matters which precise links they have, as they anticipate the (subgame perfect) equilibrium securities designed across those links as well as the resulting trades and prices of the equilibrium assets. What is more, our paper does not suffer from the indeterminacy arising in Bala and Goyal (2000) whereby the model predicts for instance that under some conditions all equilibrium networks are hub-and-spoke, but does not provide any guidance as to which of the nodes emerges as the hub.

Briefly, we derive the following results in this paper. We prove existence of equilibria. We show that network externalities give rise to networks that are suboptimal for arbitrageurs. Controlling for depth, an optimal network is a hub-spoke network. The unrestricted network, in which all links are permissible, is always suboptimal. If the unrestricted network is hub-and-spoke, it uses the suboptimal hub. The reason has to do with the provision of liquidity. Roughly speaking, the optimal hub is a hub whose equilibrium state price vector lies towards the center of all nodes so as to be used as a repository of liquidity. This allows gains from trade to be reaped with as little a market impact as possible, provided all arbitrageurs use the same hub. However, each arbitrageur, if given the opportunity, has an incentive to deviate and form a link across two exchanges, one on each side of the hub, since such a link represents larger gains from trade. The deviating player will therefore not only not contribute to liquidity, but will in fact use up liquidity at both ends. All other players act similarly, leading to a Prisoner's Dilemma style inefficient outcome.

We explicitly characterize the equilibrium asset structure which essentially consists of a portfolio of swaps between various sets of autarky state prices, with the swap rates depending on the network.

As the number of arbitrageurs converges to infinity, all equilibrium state prices of exchanges belonging to a connectable subset (i.e. exchanges that can be linked directly or indirectly) must converge to the frictionless complete-markets Walrasian state price vector of the integrated economy consisting of exchanges in that subset. In that sense, arbitrageurs connect markets and ensure securities trades in aggregate that exactly coincide with the transfers of securities that a global Walrasian auctioneer would have performed. This is true despite the inefficiencies arising from the network externalities, from market power and from the fact that any arbitrageur is only allowed to connect two exchanges. We do, however, provide an example in which the equilibrium network is not connected, even though it is connectable.

A note on assumptions:
We model each trading location or exchange as a standard GEI (general equilibrium with incomplete markets) economy. Arbitrageurs take the Walrasian demand function on each exchange as given and play a Cournot trading game in asset supplies (for a fixed asset structure). In order to characterize the Cournot-Walras equilibria of this game, we assume that the Walrasian demand functions are linear in asset

supplies. More precisely, we assume that state prices on an exchange are linear in the net aggregate endowment (aggregate endowment plus asset supplies) of the exchange, i.e. the CAPM holds with respect to net aggregate endowments. This CAPM assumption suffices for our results on security design and network structure. One way to deliver the CAPM is to assume quadratic utility. This is what we do at the outset.

2 The Setup

We consider a two-period economy with uncertainty parametrized by the state space $S := \{1, \ldots, S\}$. Assets are traded in several locations or "exchanges." They are in zero net supply. We do not impose complete markets or the existence of a riskless asset.

Investor $i \in I^k := \{1, \ldots, I^k\}$ on exchange $k \in K := \{0, \ldots, K\}$ has endowments $(\omega_0^{k,i}, \omega^{k,i}) \in \mathbb{R} \times \mathbb{R}^S$, and preferences which allow a quasilinear quadratic representation

$$U^{k,i}(x_0^{k,i}, x^{k,i}) = x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{1}{2} \beta^{k,i} (x_s^{k,i})^2 \right],$$

where $x_0^{k,i} \in \mathbb{R}$ is consumption at date 0, $x^{k,i} \in \mathbb{R}^S$ is consumption at date 1, and π_s is the probability (common across agents) of state s. The coefficient $\beta^{k,i}$ is positive. Investors are price-taking and can trade only on their own exchange.

In addition there are arbitrageurs who possess the trading technology which allows them to also trade across exchanges. For simplicity, we assume that arbitrageurs only care about time zero consumption. They are imperfectly competitive.

Asset payoffs on exchange k are given by a full column rank payoff matrix R^k of dimension $S \times J^k$. The asset span on exchange k is the column space of R^k, which we denote by $\langle R^k \rangle$. We assume that all assets are arbitraged.[1]

Given the set of exchanges K, a network *architecture* $\mathcal{A}$ is defined by a set of admissible links (k, ℓ), i.e. $\mathcal{A} \subset \{(k, \ell) : k, \ell \in K, k \neq \ell\}$. Each arbitrageur chooses to arbitrage one of the admissible links. Let $N^{k\ell}$ be the number of arbitrageurs[2] on link (k, ℓ). We use the same notation for the set of arbitrageurs on link (k, ℓ). Clearly $N^{k\ell} = N^{\ell k}$. For notational convenience we define $N^{k\ell}$ to be zero if $k = \ell$. We have $\sum_{k, \ell > k} N^{k\ell} = N$.

[1] This is an innocuous assumption. It is straightforward to extend our analysis to the case where, on a given exchange, some assets are not arbitraged, i.e. traded only by investors on the exchange, while other assets are arbitraged, i.e. traded by both investors and arbitrageurs. It turns out, however, that equilibrium prices of arbitraged assets are not affected by the payoffs of non-arbitraged assets. Thus the characteristics of non-arbitraged assets have no bearing on arbitrage trades or on security design by arbitrageurs. Of course, these assets do affect the equilibrium allocation (see Footnote 5).

[2] It will be shown that the only equilibrium is the symmetric equilibrium wherein all arbitrageurs active on a given link act symmetrically.

We say that a network architecture is *unrestricted* if any link (k, ℓ) is admissible; if not, it is a *restricted* architecture. We will be focusing on two kinds of restricted architectures: *hub-spoke* (or star) architectures and *unary* architectures. A hub-spoke architecture with exchange k as the hub is given by $\{(k, \ell) : \ell \in K, \ell \neq k\}$. For the sake of brevity, we will refer to the hub-spoke architecture with k as the hub as the h_k-architecture. An architecture is unary if only one link is admissible. We will refer to the unary architecture with admissible link (k, ℓ) as the $u_{k\ell}$-architecture. Of course, a general architecture could be characterized by an admissible links structure that is a union of disjoint sub-architectures. Each such sub-architecture can be studied in isolation, some of which might be unary or of the hub-and-spoke variety.

A *network* consists of an architecture together with a distribution of arbitrageurs across links that are admissible in that architecture. If $N^{k\ell} > 0$ (so that in particular $(k, \ell) \in \mathcal{A}$) we say that the link (k, ℓ) is *active.* Formally, a network is described by

$$\Big\{\mathcal{A}, \{N^{k\ell}\}_{k,\ell \in K} : N^{k\ell} = 0 \text{ for } (k, \ell) \notin \mathcal{A}, \sum_{k,\ell>k} N^{k\ell} = N\Big\}.$$

Thus, for example, a hub-spoke network is a network where the active links form a hub-spoke structure. When describing networks, it is always implicitly assumed that the network architecture is such that the network can have said structure. For instance, when speaking of an h_k-network, it is understood that the architecture $\mathcal{A}$ contains the h_k-architecture. A subnetwork with nodes $G \subset K$ is said to be *connected* if each exchange in G is directly or indirectly connected with any other exchange in G. Formally,

Definition 1 *Given a network architecture $\mathcal{A}$, a subset of exchanges $G \subset K$ is* <u>*connectable*</u> *if for each $k, \ell \in G$, there is an array $\{k_1, \ldots, k_I\}$ in G such that $k_1 = k$, $k_I = \ell$ and $(k_i, k_{i+1}) \in \mathcal{A}$, for all $i = 1, \ldots, I-1$. G is* <u>*connected*</u> *if, in addition, $N^{k_i k_{i+1}}/N > 0$,*[3] *for all $i = 1, \ldots, I-1$. G is* <u>*maximally connectable*</u> *(resp. connected) if there is no $j \in K \setminus G$ such that $G \cup \{j\}$ is connectable (resp. connected).*

Any network can have the exchanges K partitioned into a collection of maximally connectable or connected subnetworks, denoted by $\mathcal{P}_0$ and $\mathcal{P}$ respectively. The various cells of $\mathcal{P}_0$ can be thought of as separate economies, so it would be without loss of generality to assume that $\mathcal{P}_0 = \{K\}$. The partition $\mathcal{P}$ depends on $\{N^{k\ell}\}$ since this array dictates connectedness of the network. To save on notation, we write $\mathcal{P}(N)$, where the index refers to the Nth element of the sequence $\big\{N, \{N^{k\ell}(N)\}_{(k,\ell)\in\mathcal{A}}\big\}$ with $\sum_{k,\ell>k} N^{k\ell}(N) = N$. At this stage this sequence is taken as arbitrarily given, and will later be derived as part of the equilibrium.

We say that a network is an *equilibrium network*, if for the given architecture $\mathcal{A}$, $\{N^{k\ell}\}_{(k,\ell)\in\mathcal{A}}$, with $\sum_{k,\ell>k} N^{k\ell} = N$, is an equilibrium distribution of arbitrageurs,

[3] We write $N^{k_i k_{i+1}}/N > 0$ rather than $N^{k_i k_{i+1}} > 0$, since we shall be concerned with the asymptotic case where N goes to infinity, with the corresponding equilibrium distribution of arbitrageurs across links.

i.e. no arbitrageur can increase his profits by deviating to any other admissible link. In equilibrium, profits must be equal (except for integer considerations) on all active links, i.e. links on which there is some arbitrageur activity (not every admissible link is necessarily active, however). Thus we can associate an equilibrium level of profits with each architecture, and can compare this level across architectures. Likewise, we can compare equilibrium utility levels of investors across different architectures. Notice that arbitrageurs, not exchanges, determine links in equilibrium. In that sense, stability of networks (as defined in Jackson and Wolinsky (1996) for instance) is of no concern here.

3 Cournot-Walras Equilibria

We begin by studying an unrestricted network with exogenously given asset payoffs $\{R^k\}_{k\in K}$. A restricted network can be treated as a special case by exogenously fixing $N^{k\ell}=0$ for all links (k,ℓ) that are inadmissible (for example, an h_k-network is obtained by setting $N^{\ell\ell'}=0$ if ℓ and ℓ' are both distinct from k). Let $y^{k,n}_{k\ell}$ be the supply on exchange k of a typical arbitrageur n active on link (k,ℓ), $k\neq\ell$. Let $y^k_{k\ell} := \sum_{n\in N^{k\ell}} y^{k,n}_{k\ell}$ be the aggregate supply on exchange k of all arbitrageurs active on (k,ℓ), and $y^k := \sum_{k'\neq k} y^k_{kk'}$ the aggregate supply on exchange k of all arbitrageurs in the economy. Finally, let $y^{k,\backslash n}$ be the aggregate supply on exchange k of all arbitrageurs except n, i.e. $y^{k,\backslash n} = y^k - y^{k,n}_{k\ell}$.

Definition 2 *Given a network with asset structure $\{R^k\}_{k\in K}$, a Cournot-Walras equilibrium (CWE) of the economy is an array of asset price functions, asset demand functions, and arbitrageur supplies, $\{q^k:\mathbb{R}^{J^k}\to\mathbb{R}^{J^k}, \theta^{k,i}:\mathbb{R}^{J^k}\to\mathbb{R}^{J^k}, y^{k,n}_{k\ell}\in\mathbb{R}^{J^k}\}_{i\in I^k,\, n\in N^{k\ell},\, k,\ell\in K}$, such that*

i. Investor optimization: For given q^k, $\theta^{k,i}(q^k)$ solves

$$\max_{\theta^{k,i}\in\mathbb{R}^{J^k}} x^{k,i}_0 + \sum_{s\in S}\pi_s\left[x^{k,i}_s - \frac{\beta^{k,i}}{2}(x^{k,i}_s)^2\right]$$
$$s.t.\ \ x^{k,i}_0 = \omega^{k,i}_0 - q^k\cdot\theta^{k,i}$$
$$x^{k,i} = \omega^{k,i} + R^k\theta^{k,i}.$$

ii. Arbitrageur optimization: For given $q^k(y^k), q^\ell(y^\ell), y^{k,\backslash n}$, and $y^{\ell,\backslash n}$, $(y^{k,n}_{k\ell}, y^{\ell,n}_{k\ell})$ solves

$$\max_{y^{k,n}_{k\ell}\in\mathbb{R}^{J^k},\, y^{\ell,n}_{k\ell}\in\mathbb{R}^{J^\ell}} {y^{k,n}_{k\ell}}^\top q^k\left(y^{k,n}_{k\ell}+y^{k,\backslash n}\right) + {y^{\ell,n}_{k\ell}}^\top q^\ell\left(y^{\ell,n}_{k\ell}+y^{\ell,\backslash n}\right)$$

$$s.t.\ \ R^k y^{k,n}_{k\ell} + R^\ell y^{\ell,n}_{k\ell} \le 0.$$

iii. Market clearing:

$$\sum_{i \in I^k} \theta^{k,i}(q^k(y^k)) = y^k, \qquad \forall k \in K.$$

Note that investors take asset prices as given, while arbitrageurs compete Cournot-style. This equilibrium concept is due to Gabszewicz and Vial (1972), and a review can be found in Mas-Colell (1982). Arbitrageurs maximize time zero consumption, i.e. profits from their arbitrage trades, but subject to the restriction that they are not allowed to default in any state at date 1. Equivalently, arbitrageurs need to be completely collateralized. Arbitrageurs have no endowments.

Let $\Pi :=$ diag $(\pi_1, \ldots, \pi_S)$ and $\mathbf{1} := (1 \ldots 1)^\top$. Investor (k,i)'s utility can be written as

$$U^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} + \mathbf{1}^\top \Pi(\omega^{k,i} + R^k\theta^{k,i}) - \frac{\beta^{k,i}}{2}(\omega^{k,i} + R^k\theta^{k,i})^\top \Pi(\omega^{k,i} + R^k\theta^{k,i}). \quad (1)$$

The first order condition for the investor's optimization problem gives us his asset demand function:

$$\theta^{k,i}(q^k) = \frac{1}{\beta^{k,i}}({R^k}^\top \Pi R^k)^{-1}[{R^k}^\top \Pi p^{k,i} - q^k], \qquad (2)$$

where $p^{k,i} := (\mathbf{1} - \beta^{k,i}\omega^{k,i})$ is agent (k,i)'s no-trade state price deflator or pricing kernel.[4] We can now use the market clearing condition to deduce the inverse demand mapping, i.e. the price vector on exchange k that sets aggregate demand, $\theta^k := \sum_{i \in I^k} \theta^{k,i}$, equal to aggregate arbitrage supply y^k:

$$q^k(y^k) = {R^k}^\top \Pi[p^k - \beta^k R^k y^k], \qquad (3)$$

where $\beta^k := [\sum_i(\beta^{k,i})^{-1}]^{-1}$, $\omega^k := \sum_i \omega^{k,i}$, and $p^k := \mathbf{1} - \beta^k\omega^k$. p^k is exchange k's autarky state price deflator (autarky with respect to the rest of the world, but allowing trade within k). We assume that $p^k \geq 0$, for all $k \in K$, which says that the representative investor on each exchange is nonsatiated at the aggregate endowment point of that exchange. The parameter β^k represents the "depth" of exchange k, i.e. the price impact of a unit of arbitrageur trading. For instance, ceteris paribus, the market impact of a trade is smaller on exchanges with a larger population—it can be absorbed by more investors. Notice that we can interpret equilibrium prices as risk-neutral prices ${R^k}^\top \Pi \mathbf{1}$ from which a risk-aversion discount $\beta^k {R^k}^\top \Pi(\omega^k + R^k y^k)$ is subtracted.

The collection $\{p^k\}_{k \in K}$ as a subset of the state space $\mathbb{R}^S$ forms the nodes of the network, and these nodes define what we call an *economy.* A network architecture specifies the admissible links between the nodes, and a network specifies an architecture together with a distribution of arbitrageurs across admissible links.

[4]The vector p is a state price deflator for (q^k, R^k) if $q^k = {R^k}^\top \Pi p$.

Our assumptions on preferences, in conjunction with the absence of nonnegativity constraints on consumption, guarantee that the equilibrium pricing function on an exchange does not depend on the initial distribution of endowments, but merely on the aggregate endowment of the local investors. The autarky state price deflator p^k also does not depend on R^k, even though investors on exchange k do trade among themselves consumptions in the span of R^k.[5]

We now solve the Cournot game among arbitrageurs, given the asset price function (3). It turns out that there is a unique CWE, and that this equilibrium is symmetric, i.e. supplies of all arbitrageurs on a given link (k, ℓ) are the same.

Of all state price deflators $\tilde{p}^k$ satisfying $q^k = {R^k}^\top \Pi \tilde{p}^k$, one will turn out to be of special importance, and we denote it by $\hat{p}^k$. It is defined by

$$\hat{p}^k := p^k - \beta^k R^k y^k, \tag{4}$$

and we call it the *equilibrium state price deflator*. It provides us with a valid state price deflator as a function of y^k and R^k. Notice that the state prices are affine in net aggregate endowments $\omega^k + R^k y^k$, or equivalently that the CAPM relation holds with respect to net aggregate endowments. As mentioned before, this is in fact what we need for our results on security design and network structure; the assumption of quadratic utility is just one way to deliver it.

The following spanning condition will turn out to be important:

(S) $\hat{p}^k - \hat{p}^\ell \in \langle R^k \rangle \cap \langle R^\ell \rangle$, $\forall k, \ell \in K$ such that $N^{k\ell} \neq \varnothing$.

We will show in Section 4 that any $\{R^k\}$ satisfying **S** is an equilibrium security design. At this stage **S** involves endogenous variables. Later we provide an equivalent condition in terms of exogenous parameters.

It is convenient to state arbitrageur supplies in terms of the supply of state-contingent consumption.

Lemma 1 (Equilibrium supplies) *Given a network with asset structure $\{R^k\}$, the equilibrium supply of arbitrageur $n \in N^{k\ell}$, $N^{k\ell} \neq \varnothing$, solves*

$$R^k y^{k,n}_{k\ell} = -R^\ell y^{\ell,n}_{k\ell} = \frac{1}{\beta^k + \beta^\ell} \cdot (\hat{p}^k - \hat{p}^\ell), \tag{5}$$

provided **S** *is satisfied.*

This lemma shows why $\hat{p}^k$ is the fundamental state price deflator to consider. The Cournot-Walras equilibrium is symmetric: all arbitrageurs on a given link have the same supply (as we shall see shortly, the CWE is also unique). The interpretation of (5) is straightforward. Arbitrageurs on link (k, ℓ) supply consumption in state s to

[5] The autarky equilibrium allocation does depend on the local asset structure, however. Since agents' endowments may not be spanned, markets are not effectively complete, and the autarky equilibrium allocation is not (locally) Pareto efficient in general.

exchange k when the price that agents on exchange k are willing to pay for a unit of state s consumption exceeds the price at which arbitrageurs can procure that unit on exchange ℓ.

The factor of proportionality in (5) is determined by depth. The deeper the involved exchanges k and ℓ (i.e. the lower are β^k and β^ℓ), the more arbitrageur n trades, since he can afford to augment his supply without affecting margins as much. Notice that implicitly, as we shall see, the equilibrium gains from trade $\hat{p}^k - \hat{p}^\ell$ depend on competition as well as on all other arbitrageur trades on the respective exchanges. In particular, we shall see that the supply vector is scaled to zero as competition intensifies, because at the same time gains from trade shrink and there are more players to share the smaller pie with.

We can solve for the equilibrium pricing kernels $\hat{p}^k$, $k \in K$, as follows. From (5),

$$R^k y^k = \sum_\ell N^{k\ell} R^k y^{k,n}_{k\ell} = \sum_\ell \frac{N^{k\ell}}{\beta^k + \beta^\ell} (\hat{p}^k - \hat{p}^\ell). \tag{6}$$

Let $\alpha^{k\ell} := \frac{N^{k\ell}}{\beta^k+\beta^\ell}$, and $\alpha^k := \sum_\ell \alpha^{k\ell}$. Using (4), we get:

Lemma 2 (Equilibrium prices: general case) *Given a network with asset structure $\{R^k\}$, consider a solution $\{\hat{p}^k\}_{k\in K}$ to the following system of equations:*

$$(1 + \beta^k \alpha^k)\hat{p}^k - \beta^k \sum_\ell \alpha^{k\ell} \hat{p}^\ell = p^k, \qquad k \in K. \tag{7}$$

If $\mathbf{S}$ *is satisfied at that $\{\hat{p}^k\}_{k\in K}$, then $\{\hat{p}^k\}_{k\in K}$ is a profile of equilibrium pricing kernels.*

Define the matrix $\underline{m}$ as $\underline{m}_{kk} := 1 + \beta^k \alpha^k$ and $\underline{m}_{k\ell} = -\beta^k \alpha^{k\ell}$ for $k \neq \ell$. Define the matrix M as $\underline{m} \otimes I_{S\times S}$. Then (7) can be written, using $\hat{p} := \{\hat{p}^k\}_{k\in K}$ and $p := \{p^k\}_{k\in K}$, as $M\hat{p} = p$.

Lemma 3 *The matrix M is nonsingular. Hence, there exists a unique $\hat{p}$ solving $M\hat{p} = p$, namely $\hat{p} = M^{-1}p$. Moreover $M^{-1} \geq 0$, so that $\hat{p} \geq 0$.*

The equilibrium state price vector on any $k \in K$ is a nonnegative linear combination of all autarky state prices, p^ℓ, $\ell \in K$. How much p^ℓ is impounded into $\hat{p}^k$ depends on depths as well as on $\{N^{k\ell}\}$. Intuitively, if in equilibrium state prices on k depend on preferences and endowments on other exchanges, it is due to the arbitrage trades which integrate the various exchanges. The higher a particular $N^{k\ell}$ and the lower β^k and β^ℓ, the more arbitrageurs transfer state-contingent consumption in equilibrium across k and ℓ, thereby reducing outstanding gains from trade $\hat{p}^k - \hat{p}^\ell$, increasing the influence of preferences and endowments (and hence of p^ℓ) of other exchanges ℓ on local state prices $\hat{p}^k$. On top of that, given that yet other arbitrageurs transfer resources between $j \neq \ell, k$ and ℓ, the state prices of j also find their way into $\hat{p}^k$.

This explains why, depending on $\{N^{k\ell}\}$, autarky state prices of all exchanges may be reflected in each one of them in equilibrium.

Notice that we can rewrite $\hat{p} = M^{-1}p = (\underline{m}^{-1} \otimes I_{S\times S})p$, and so $\hat{p}^k = p^{\eta,k} := \sum_{j\in K} \eta^{kj} p^j$, where η^{kj} is the element (k,j) of $\underline{m}^{-1}$. Since $\underline{m}\mathbf{1} = \mathbf{1}$ we also have $\underline{m}^{-1}\mathbf{1} = \mathbf{1}$. It follows that the equilibrium state vector on any exchange k is the average of all autarky state vectors $\{p^j\}$, with weights $\eta^{kj} \geq 0$, $\sum_{j\in K} \eta^{kj} = 1$. The weights are endogenous and depend on the extent to which the various exchanges are interrelated in equilibrium.

Lemma 3 allows us to rewrite condition **S** directly in terms of exogenous parameters:

(S) $p^{\eta,k} - p^{\eta,\ell} \in \langle R^k \rangle \cap \langle R^\ell \rangle$, $\forall k, \ell \in K$ such that $N^{k\ell} \neq \varnothing$.

A crucial and obvious consequence of (5) and (7) which we shall use repeatedly in the sequel is:

Corollary 1 *$\{\hat{p}^k, y_{k\ell}^{k,n}\}$ does not depend on $\{R^k\}$, provided* **S** *holds.*

A question of considerable interest are the characteristics of the equilibrium that occurs in the limit as the number of arbitrageurs is raised towards infinity, and in particular whether equilibria (under **S**) in our model can be approximated by complete markets Walrasian equilibria, with all the resulting welfare implications.

Let $G \in \mathcal{P}(\infty)$.

$$p_G^\lambda := \sum_{k\in G} \lambda_G^k p^k$$

where

$$\lambda_G^k := \frac{\frac{1}{\beta^k}}{\sum_{j\in G} \frac{1}{\beta^j}}.$$

The vector p_G^λ is the average willingness to pay of all investors in G, with the willingness to pay on each exchange weighted by its relative depth. p_G^λ is the pricing kernel for the complete-markets Walrasian equilibrium (with unrestricted participation) when the economy is only composed of exchanges in G, see Rahi and Zigrand (2005). If $G = K$, then $p^\lambda := p_K^\lambda$ is the pricing kernel for the global complete-markets Walrasian equilibrium (with unrestricted participation).

Lemma 4 (Convergence) *For any k, ℓ in $G \in \mathcal{P}(\infty)$, $\hat{p}^k(\infty) = \hat{p}^\ell(\infty) = p_G^\lambda$, where $\hat{p}^i(\infty) := \lim_{N\to\infty} \hat{p}^i(N)$, provided* **S** *holds.*

As the number of arbitrageurs increases without bound, all mispricings across exchanges in the same connected subset G of exchanges vanish. The resulting equilibrium pricing converges to the Walrasian one of an economy comprised of exchanges in G only. Should the $N^{k\ell}$, as N goes to infinity, increase in a way that connects all K exchanges in the limit, then equilibrium state prices converge to complete markets

Walrasian state prices of the entire initial economy: arbitrageurs connect markets and ensure securities trades in aggregate that exactly coincide with the transfers of securities that a global Walrasian auctioneer would have performed. All that is required is that **S** holds, whether markets are complete or not, and we show below that **S** does hold in equilibrium when securities are designed. Of course, ultimately $N^{k\ell}$ as a function of N is determined endogenously by the equilibrium distribution of arbitrageurs across links, and we shall return to convergence when characterizing the distribution as an outcome of the network formation game.

While Lemma 3 gives us an explicit solution to (7), the solution for the general case is unwieldy and difficult to manipulate analytically. We therefore derive below the closed-form solutions for a hub-spoke network and for the case of $K = 2$ only.

Consider first a hub-spoke network, say with exchange 0 as the hub. While there is only one h_0-architecture, there are many h_0-networks, depending on the distribution $\{N^{0k}\}$. Let

$$p^\gamma := \sum_{k \in K} \gamma^k p^k$$

where

$$\gamma^k := \frac{\frac{\beta^0 \alpha^{0k}}{1+\beta^k \alpha^{0k}}}{1 + \beta^0 \sum_j \frac{\alpha^{0j}}{1+\beta^j \alpha^{0j}}}, \qquad k \neq 0$$

$$\gamma^0 := \frac{1}{1 + \beta^0 \sum_j \frac{\alpha^{0j}}{1+\beta^j \alpha^{0j}}}.$$

Lemma 5 (Equilibrium prices: hub-spoke network) *Consider an h_0-network with given $\{R^k\}$, and suppose* **S** *holds. Then $\hat{p}^0 = p^\gamma$, and*

$$\hat{p}^k = (1 + \beta^k \alpha^{0k})^{-1}(p^k + \beta^k \alpha^{0k} p^\gamma), \qquad k \neq 0. \tag{8}$$

Moreover, for an h_0-network, **S** *is equivalent to the following condition:*

($\mathbf{S^{h_0}}$) $p^k - p^\gamma \in \langle R^k \rangle \cap \langle R^0 \rangle$, $\forall k \neq 0$.

Note that p^γ depends on $\{N^{0k}\}$ but not on $\{R^k\}$. Hence, for a given h_0-network, the pricing functional induced by p^γ on $\langle R^0 \rangle$ can be extended to the whole space $\mathbb{R}^S$. Using the convergence result 4, we see that equilibrium valuation and allocation on each exchange converge to the complete-markets Walrasian ones.

The weights used for netting the gains from trade are different since trades between exchanges k and ℓ are not mediated by the same set of arbitrageurs. The exchanges are connected nevertheless through exchange 0 on which all arbitrageurs trade. Note that, unlike for λ^j, the weight γ^j is increasing in N^{0j}, the number of arbitrageurs active on exchange j, and γ^j converges to λ^j as N^{0j} goes to infinity for every j. In other words, even though no single arbitrageur ties all the markets together, fierce competition is a substitute for unrestricted access to global markets.

Of course, if on the other hand $N^{0k} = 0$, the exchange k is irrelevant and does not influence any of the results.

Intuitively, since arbitrageurs who trade across $\{0, k\}$, $k \neq 0$, do not trade anywhere else, one may have expected to see that consumption supply is proportional to $p^k - p^0$, the autarky gains from trade between k and 0. But this would neglect the fact that the gains between k and 0 also depend on the trade of all other arbitrageurs $\{0, j\}$, $j \notin \{0, k\}$, with 0.

Now consider the case of three exchanges ($K = 2$). Let

$$\nu := \left(\beta^0\beta^1 + \beta^0\beta^2 + \beta^1\beta^2\right)\left(\alpha^{01}\alpha^{02} + \alpha^{01}\alpha^{12} + \alpha^{02}\alpha^{12}\right)$$

The equilibrium pricing kernels are given in the following lemma.

Lemma 6 (Equilibrium prices: three exchanges) *Suppose $K = \{0, 1, 2\}$. Consider a network with asset structure $\{R^k\}$ that satisfies* **S**. *Then the equilibrium pricing kernels are*

$$\hat{p}^k = \frac{1}{1 + N + \nu} \cdot \left[(1 + N - \beta^k\alpha^k)p^k + \beta^k\alpha^{k\ell}p^\ell + \beta^k\alpha^{k\ell'}p^{\ell'} + \nu p^\lambda\right]$$

for $\ell \neq \ell'$ distinct from k.

It is straightforward to verify that the $\{\hat{p}^k\}$ given in the lemma satisfy (7).

Finally, we calculate equilibrium arbitrageur profits. We will be using this information in the sequel. We will need to distinguish between equilibrium arbitrageur profits for a given network (in particular, for a given distribution of arbitrageurs $\{N^{k\ell}\}$), and equilibrium profits in an equilibrium network (with endogenously determined $\{N^{k\ell}\}$). Let $\varphi^{k\ell}$ be the equilibrium level of profits on link (k, ℓ) for given $\{N^{k\ell}\}$. In an equilibrium network, let Φ, Φ^{h_k} and $\Phi^{u_{k\ell}}$ denote profits in the unrestricted network, the h_k-network, and the $u_{k\ell}$-network respectively. For state-contingent consumption $x \in \mathbb{R}^S$, the $L^2(\Pi)$-norm of x is defined as follows: $\|x\|_2 := (x^\top \Pi x)^{\frac{1}{2}}$.

Lemma 7 (Equilibrium profits) *Consider a network with asset structure $\{R^k\}$ that satisfies* **S**. *Then, for $N^{k\ell} \neq \varnothing$, the equilibrium profit of arbitrageur $n \in N^{k\ell}$ is*

$$\varphi^{k\ell} = \frac{1}{\beta^k + \beta^\ell} \cdot \|\hat{p}^k - \hat{p}^\ell\|_2^2. \tag{9}$$

In an h_0-network:

$$\varphi^{0k} = \frac{\beta^k + \beta^0}{[(1 + N^{0k})\beta^k + \beta^0]^2} \cdot \|p^k - p^\gamma\|_2^2. \tag{10}$$

For the case of three exchanges, $K = \{0, 1, 2\}$:

$$\begin{aligned}\varphi^{k\ell} = {} & \frac{1}{(1 + N + \nu)^2(\beta^k + \beta^\ell)} \cdot \\ & \Big[(1 + N - N^{k\ell} - \beta^k\alpha^{k\ell'})(1 + N - N^{k\ell} - \beta^\ell\alpha^{\ell\ell'})\|p^k - p^\ell\|_2^2 \\ & \quad + (\beta^\ell\alpha^{\ell\ell'} - \beta^k\alpha^{k\ell'})(1 + N - N^{k\ell} - \beta^k\alpha^{k\ell'})\|p^k - p^{\ell'}\|_2^2 \\ & \quad - (\beta^\ell\alpha^{\ell\ell'} - \beta^k\alpha^{k\ell'})(1 + N - N^{k\ell} - \beta^\ell\alpha^{\ell\ell'})\|p^\ell - p^{\ell'}\|_2^2\Big], \qquad \ell' \neq k, \ell.\end{aligned} \tag{11}$$

Finally, in the $u_{k\ell}$-network:

$$\varphi^{k\ell} = \Phi^{u_{k\ell}} = \frac{1}{(1+N)^2(\beta^k + \beta^\ell)} \cdot \|p^k - p^\ell\|_2^2. \tag{12}$$

Note that there is only one $u_{k\ell}$-network, since all N arbitrageurs must be on link (k, ℓ).

4 Security Design by Arbitrageurs

We have seen that there is a unique CWE associated with any asset structure $\{R^k\}_{k\in K}$ and distribution of arbitrageurs $\{N^{k\ell}\}_{k\in K}$. In this section we endogenize the security payoffs, keeping the distribution of arbitrageurs fixed (the latter is endogenized in the next section). Arbitrageurs play a security design game the outcome of which is an equilibrium asset structure. The payoffs of arbitrageurs are the profits they earn in the CWE associated with this asset structure. Arbitrageurs on link (k, ℓ) choose the asset payoffs R^k and R^ℓ. The asset structure $\{R^k\}$ is a Nash equilibrium of the security design game if no arbitrageur stands to gain by introducing additional assets that he may trade monopolistically (clearly this is also a Nash equilibrium of the associated game in which all arbitrageurs trade the additional securities). An asset structure is optimal for an arbitrageur if it yields the highest profits for the arbitrageur in the associated CWE, among all possible asset structures. An asset structure is Pareto optimal for arbitrageurs if there is no alternative asset structure that Pareto dominates it in equilibrium for the arbitrageurs.

Proposition 1 (Security design: general case) *For a given network, any asset structure $\{R^k\}$ satisfying* **S** *is a Nash equilibrium of the security design game. Therefore the single security with payoff*

$$R^k = R^\ell = p^{\eta,k} - p^{\eta,\ell}, \qquad k, \ell \in K, N^{k\ell} \neq 0 \tag{13}$$

issued by any of the $N^{k\ell}$ arbitrageurs to both k and ℓ is a Nash equilibrium of the security design game.

In particular, the complete asset structure $R^k = I_{S\times S}$, for all k, is a Nash equilibrium. Among all equilibrium asset structures, a minimal asset structure is one with the smallest number of assets. By Corollary 1, all asset structures satisfying **S** are payoff-equivalent for arbitrageurs. A minimal asset structure in this class would be the one chosen if each security issued bore a fixed cost c, no matter how small.[6] A minimal security design can be obtained from the characterization (13) by removing (locally) redundant assets on each exchange. Notice that $\sum_{j\in K}(\underline{m}_{kj}^{-1} - \underline{m}_{\ell j}^{-1}) = 0$.

A much crisper image appears in the hub-spoke constellation:

[6] In fact, such fixed costs are significant; see Tufano (1989) for an empirical assessment.

Proposition 2 (Security design: hub-spoke network) *In an h_0-network, any asset structure satisfying* $\mathbf{S^{h_0}}$ *is a Nash equilibrium of the security design game. The security design*

$$R^k = p^k - p^\gamma, \qquad k \in \{1, \ldots, K\} \tag{14}$$

with R^0 a full rank matrix such that $p^k - p^\gamma \in \langle R^0 \rangle$, for all $k \neq 0$, is a minimal Nash equilibrium of the security design game.

In the limiting case where N^{0k} tends to infinity for all k, this security design is $\{p^k - p^\lambda\}$, which is the minimal Nash equilibrium in the universal arbitrage scenario (Rahi and Zigrand (2005)). Notice also that at the equilibrium condition $\mathbf{S^{h_0}}$, and therefore $\mathbf{S}$, necessarily holds, and can be dropped without loss of generality.

For a unary network, we have an even sharper characterization. Since there is only a single active link, the two cases of universal and restricted arbitrage coincide. The following result is immediate from Rahi and Zigrand (2005):

Proposition 3 (Security design: unary network) *Consider the $u_{k\ell}$-network. The following statements are equivalent:*

i. Condition $\mathbf{S}$ *holds.*

ii. $p^k - p^\ell \in \langle R^k \rangle \cap \langle R^\ell \rangle$;

iii. the asset structure $\{R^k, R^\ell\}$ is optimal for arbitrageurs;

iv. the asset structure $\{R^k, R^\ell\}$ is a Nash equilibrium of the security design game.

Thus the complete asset structure $R^1 = I_{S \times S}$ is optimal for arbitrageurs, and a Nash equilibrium. Again, notice that for the $u_{k\ell}$-network, at an equilibrium condition $\mathbf{S}$ necessarily holds, and can be dropped without loss of generality. Moreover, all optimal/equilibrium asset structures are payoff-equivalent for arbitrageurs. These asset structures span the net trades between the two exchanges in the complete-markets Walrasian equilibrium. We have the following corollary (when we say "unique", we mean "unique up to scaling").

Proposition 4 *Consider the $u_{k\ell}$-network. The asset structure $R^k = R^\ell = p^k - p^\ell$ is*

i. the unique minimal optimal asset structure for arbitrageurs; and

ii. the unique minimal Nash equilibrium of the security design game.

The minimal optimal security is the difference of the autarky state price deflators of the two exchanges.

Our analysis (for the general case) readily extends to the more realistic case where arbitrageurs can introduce new assets, but they cannot affect the payoffs of the existing assets $\{R^k\}$. In other words, security design really represents incremental "innovation." The following result is immediate from Propositions 1 and 2.

Proposition 5 (Innovation) *Given a network with initial asset structure* $\{R^k\}$*, the asset structure*

$$\begin{array}{ccc} \left[R^k \quad p^{\eta,k} - p^{\eta,\ell}\right] & \textit{if} & p^{\eta,k} - p^{\eta,\ell} \notin \langle R^k \rangle, \\ R^k & \textit{if} & p^{\eta,k} - p^{\eta,\ell} \in \langle R^k \rangle, \end{array}$$

is a minimal Nash equilibrium of the security design game.

Since for a given $\{R^k, N^{k\ell}\}$ arbitrageurs on link (k, ℓ) find it optimal to supply state-contingent consumption proportional to $p^{\eta,k} - p^{\eta,\ell}$ if allowed, they innovate on exchange k if and only if $p^{\eta,k} - p^{\eta,\ell} \notin \langle R^k \rangle$, in which case they "further complete" the market by adding $p^{\eta,k} - p^{\eta,\ell}$. Equivalently, they could add a security that makes $p^{\eta,k} - p^{\eta,\ell}$ tradable in conjunction with R^k.

5 Equilibrium Networks

Before we specialize the problem to isolate the interesting scale effects, interdependencies, externalities and tradeoffs in arbitrage networks, we report on some results that apply generally. We are able to show the following endogenization (the $N^{k\ell}$ are now functions of N determined at the equilibrium of the network formation game) of Lemma 4. It says that as $N \to \infty$, all state prices within a maximal connected subnetwork architecture converge to a common one.

Lemma 8 (Equilibrium Convergence) *Assume an arbitrary network architecture. Given any* $G_0 \in \mathcal{P}_0$*,* $\lim_{N\to\infty} \hat{p}^k_{G_0} = p^\lambda_{G_0}$*,* $k \in G_0$*.*

Notice that since $\mathcal{P}_0$ is coarser than $\mathcal{P}(N)$, any $G_0 \in \mathcal{P}_0$ is a union of elements in $\mathcal{P}(N)$. So for any G, G' in $\mathcal{P}(\infty)$ with $G, G' \subset G_0 \in \mathcal{P}_0$, we have $\lim_{N\to\infty} \hat{p}^k_G = p^\lambda_G = p^\lambda_{G'} = \lim_{N\to\infty} \hat{p}^k_{G'}$. Alternatively, if for two arbitrary subsets of some $G_0 \in \mathcal{P}_0$, K_1 and K_2, we have $p^\lambda_{K_1} \neq p^\lambda_{K_2}$, then K_1 and K_2 must be connected for large enough N. As stated previously, it would be without loss of generality to assume that $\mathcal{P}_0 = \{K\}$, so that $\lim_{N\to\infty} \hat{p}^k = p^\lambda$. So either all connectable exchanges do converge to a common state price deflator as a result of the equilibrium interexchange flows of funds resulting from the equilibrium $\{N^{k\ell}(N)\}$, or if in equilibrium there emerge two disconnected subnetworks (with a connection allowed by $\mathcal{A}$ but not arising in equilibrium), then even though there are no flows between them, the equilibrium state price vector on each one of them must independently converge to a common state price in the limit. We shall provide such an example in the sequel.

There are cases where, for small N, not all admissible links are active because each arbitrageur can only arbitrage across a single link, and would therefore choose the most attractive opportunities first. But as $N \to \infty$, ultimately each exchange k will see some arbitrage trade, as long as there is some reward to be reaped, i.e. as long as the autarky state price is not equal to $p^\lambda_{G_0}$, $k \in G_0$.

When analyzing the overall equilibrium of this multiple stage game, in this paper we focus on equilibria whose equilibria in the design stage satisfy **S**. We have shown that there always is such a subgame perfect equilibrium. For a given economy and a subgame perfect equilibrium with three arbitrary exchanges satisfying **S**, we are now able to ask the question as to which network architecture maximizes profits for arbitrageurs. In principle, this boils down to comparing the level of profits (11), evaluated at the equilibrium arbitrageur distribution (which is determined by the condition that $\varphi^{k\ell}$ is equal across all active links (k, ℓ)), for each candidate network architecture. This combinatorial problem, as one might expect, leads to very few clear-cut general results since so many tradeoffs must be balanced, such as the various depths, the various initial mispricings across all active links, taking into account that the prices on each exchange may depend on all flows across the network, no matter how "remote." In particular, one should not expect to derive general results of the sort "all Nash equilibrium networks are of the star form," as have been derived in Bala and Goyal (2000), for in our paper nodes are exchanges with heterogeneous intrinsic characteristics. Over and above the connectivity structure, location (in $\mathbb{R}^S$) and depth matter as well. Therefore any given connectivity structure $\{N^{k\ell}\}$, say a star, can be perturbed by varying the fundamental parameters of preferences and endowments, and therefore by scaling depth and the gains from trade resulting from the position of the autarky state prices. We proceed as follows. The next section studies some results related to the scale of depths and gains from trade, abstracting away the network externalities. Section 7 then focuses on the network effects for normalized scales.

6 Pure Scale Effects in Networks

We now endogenize the network. More precisely, for a given network architecture $\mathcal{A}$, we endogenize the concentration of arbitrageur activity $\{N^{k\ell}\}_{(k,\ell)\in\mathcal{A}}$. In conjunction with the results of the previous section, this will give us an equilibrium $\{R^k, N^{k\ell}\}$, for a given network architecture. Since an equilibrium asset structure $\{R^k\}$ depends on the arbitrageur distribution $\{N^{k\ell}\}$, an arbitrageur contemplating moving from one link to another would have to account for the corresponding change in equilibrium asset structure. Fortunately, there is an easy way to avoid this complication, by simply assuming that markets are complete on each admissible link. This is an equilibrium asset structure regardless of the distribution of arbitrageurs. Then, after determining the equilibrium distribution, we can apply the security design results of the previous section for that distribution.

In general, arbitrageur activity on one link exerts externalities, which may be positive or negative, on arbitrageurs on other links. This is an interesting feature of our model, and we will expand on it later. But it is instructive to begin by studying the class of unary networks, in which there is only one link, and therefore no externalities across links. This provides us with a useful benchmark, and illuminates the tradeoffs involved in the choice of link by an arbitrageur.

Proposition 6 *The optimal unary network architecture for arbitrageurs is $u_{k^*\ell^*}$, where*

$$(k^*, \ell^*) = \arg\max_{k,\ell} \frac{1}{\beta^k + \beta^\ell} \cdot \|p^k - p^\ell\|_2^2. \quad (15)$$

In particular, if $\beta^k = \beta^\ell$ for all k, ℓ, then

$$(k^*, \ell^*) = \arg\max_{k,\ell} \|\omega^k - \omega^\ell\|_2^2.$$

If $\omega^k = \omega^\ell$ for all k, ℓ, then

$$(k^*, \ell^*) = \arg\max_{k,\ell} \frac{(\beta^k - \beta^\ell)^2}{\beta^k + \beta^\ell}. \quad (16)$$

This proposition is immediate from (12). It is reminiscent of a result in Duffie and Jackson (1989) which says that the volume-maximizing futures contract maximizes the "endowment differential" of the long and short sides of the market.

To develop some intuition consider the case of identical endowments. Then the exchanges $k \in K$ differ only with respect to their preference parameters $\{\beta^k\}$. The function $f(\beta^k, \beta^\ell) := \frac{(\beta^k - \beta^\ell)^2}{\beta^k + \beta^\ell}$ is depicted in Figure 1 for fixed β^ℓ.

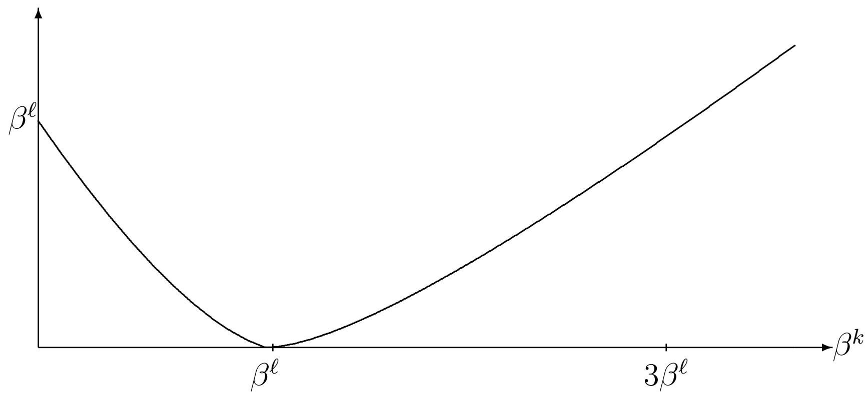

Figure 1: **The objective function of arbitrageurs**

We see that the slope of f to the left of β^ℓ is steeper than the slope to its right. The solution is to choose the exchange with the highest β^k, provided $\beta^k > 3\beta^\ell$. If $\beta^k \leq 3\beta^\ell$ for all k, then it may be optimal to choose the exchange with the lowest β^k, provided it is close enough to zero. The reason why the optimal exchange will be either the k with the smallest or the largest β^k is as follows. The term $(\beta^k - \beta^\ell)^2$ measures the extent of the unutilized gains from trade between exchanges k and ℓ. These gains are largest the furthest from β^ℓ the new exchange is located. But β^k also determines the shallowness of the exchange. This is reflected in the denominator of the expression in (16). The slope to the left of β^ℓ is steeper since markets with low β^k are deeper, and therefore more attractive to arbitrageurs. Thus there is a tradeoff between gains from trade and depth. The best exchange may not be the deepest one, since the restriction $\beta^k > 0$ limits the gains from trade. The shallowest exchange may be preferred to the deepest one if the gains from trade are sufficiently large to compensate for the shallowness.

Letting

$$\mu_{k\ell} := \frac{1}{\beta^k + \beta^\ell} \cdot \|p^k - p^\ell\|_2^2 ,$$

we can restate Proposition 6 as follows: the profit-maximizing link is the one that maximizes $\mu_{k\ell}$. We can interpret $\mu_{k\ell}$ as a measure of the autarky gains from trade on link (k,ℓ). For an arbitrary network, the following conjecture suggests itself: $N^{k\ell} > N^{k'\ell'}$ if and only if $\mu_{k\ell} > \mu_{k'\ell'}$. However, it turns out that this is not true in general. Even for hub-spoke networks, the conjecture holds only in the case of two spokes; with more than two, the relative position of the spokes matters in addition to the gains from trade between a given spoke and the hub.

Consider first the general hub-spoke case. Arbitrageur profits are given by (10). Ignoring integer constraints on the number of arbitrageurs, in equilibrium (with endogenous $\{N^{0k}\}$) we must have $\varphi^{0k} = \Phi$, for all k on which there is some arbitrage activity. The profit φ^{0k} is a product of two terms. The first term is decreasing in β^k, and therefore increasing in the depth of exchange k, while the second term captures the net gains from trade on exchange k. Consequently, there is a tradeoff between depth and gains from trade. The following result is immediate from an inspection of (10):

Proposition 7 (Equalizing differences: hub-spoke network) *Consider a h_0-network architecture, for which $\mathbf{S^{h_0}}$ is satisfied. Then:*

i. If $N^{0k} = N^{0\ell} > 0$, then

$$\|p^k - p^\gamma\|_2^2 > \|p^\ell - p^\gamma\|_2^2 \qquad \textit{iff} \qquad \beta^k > \beta^\ell$$

ii. If $\|p^k - p^\gamma\|_2^2 = \|p^\ell - p^\gamma\|_2^2$, then

$$N^{0k} > N^{0\ell} \qquad \textit{iff} \qquad \beta^k < \beta^\ell$$

iii. If $\beta^k = \beta^\ell$, then

$$N^{0k} > N^{0\ell} \qquad \textit{iff} \qquad \|p^k - p^\gamma\|_2^2 > \|p^\ell - p^\gamma\|_2^2$$

Proposition 7 does not give us an explicit characterization of $\{N^{0k}\}$ since p^γ itself depends on $\{N^{0k}\}$. The case of two spokes, however, is amenable to further analysis:

Proposition 8 *Suppose $K = \{0, 1, 2\}$, and $\mathbf{S}$ is satisfied. In the equilibrium of the game based upon the h_0-network architecture, $N^{01} = N^{02}$ iff $\mu_{01} = \mu_{02}$. Also, if there exists a solution $x \in [0, N]$ to (23), $N^{01} > N^{02}$ iff $\mu_{01} > \mu_{02}$ (regardless of μ_{12}).*

The proposition verifies our conjecture that in a hub-spoke network with two spokes, arbitrageur activity is higher on the spoke for which the gains from trade with the hub, as measured by μ_{0k}, are greater.

For some further results on the equilibrium distribution of arbitrageur activity, see Proposition 9.

7 Network Effects

In order to spell out the essence of the network effects, we would like to provide explicit analytical results. Unfortunately, only the three-exchange case is analytically tractable. Of course, we do have closed form solutions for arbitrageur profits for arbitrary K, so a numerical analysis is certainly feasible. But we do not attempt this here. Accordingly, we assume in much of this section that there are only three exchanges: $K = \{0, 1, 2\}$. This case lends itself to a simple taxonomy of network architectures (and therefore of networks). A network architecture is either unrestricted (three admissible links), hub-spoke (two admissible links), or unary (one admissible link). While there is only one unrestricted network architecture, there are three hub-spoke architectures, depending on the choice of hub, and three unary architectures, depending on the choice of admissible link. We assume that $N \geq 3$, for otherwise we cannot have activity on all three links. Apart from this assumption, we ignore integer constraints on the distribution of arbitrageurs $\{N^{k\ell}\}$. Taking them into account makes the exposition messy without leading to any new insights.

Proposition 6 tells us which unary network architecture is optimal for arbitrageurs, thereby abstracting from network effects. In this section, we characterize profit-maximizing networks and network architectures more generally, focusing on the network effects. The best way to proceed and derive crisper results is therefore to isolate the network tradeoffs in the first instance. Basically, we fix the scale of depth and of the gains from trade, and thereby ensure that scale effects do not overwhelm the network effects.

Isosceles Assumption. There are three exchanges, 0, 1 and 2, with identical depths given by $\beta := \beta^0 = \beta^1 = \beta^2$, and with gains from trade satisfying $\mu := \mu_{01} = \mu_{02}$.

This assumption eliminates the obvious scale effects as emphasized in the previous section, allowing us to focus on the structure of the network itself. An exchange is characterized completely by its autarky state price deflator, an element of $\mathbb{R}^S$. Without loss of generality, we can fix the location of p^0 and p^1 as in Figure 2, and consider the economies where p^2 is located somewhere on the semicircle between v and p^1. The location of p^2 is pinned down by $\mu_{12} \in [0, 4\mu]$. When $p^2 = v$ for instance, $\sqrt{\mu_{12}} = 2\sqrt{\mu}$, so that $\mu_{12} = 4\mu$. When $p^2 = w$ we get the equilateral case corresponding to $\mu_{12} = \mu$.[7]

[7] When we provide geometric intuition, it should be viewed as applying to the Hilbert space L^2 with the inner product $\langle x, y\rangle = E[xy]$, where as usual $\|x\|_2^2 = \langle x, x\rangle$.

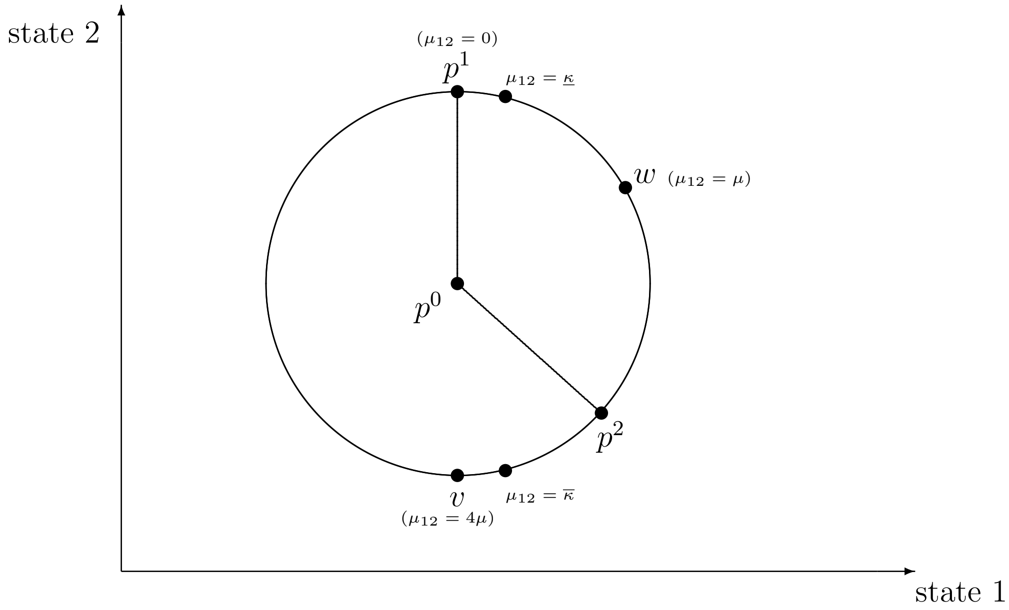

Figure 2: **An isosceles economy**

Under the Isosceles Assumption, the h_1 and h_2-architectures are symmetric, and give the same arbitrageur profits in equilibrium. Hence it suffices to consider only the h_0 and h_1-networks. Likewise, among the three unary networks, we need to look at only the u_{01} and u_{12}-networks.

Let

$$\underline{\kappa} := \frac{(N+4)^2\mu}{7N^2+20N+16}, \qquad \overline{\kappa} := \frac{4(N+1)^2\mu}{N^2+2N+4}.$$

Note that $0 < \underline{\kappa} < \mu < \overline{\kappa} < 4\mu$ (see Figure 2). $\underline{\kappa}$ is decreasing in N, and $\overline{\kappa}$ is increasing in N. At $N = 3$, $\underline{\kappa} \approx .4\mu$ and $\overline{\kappa} \approx 3.4\mu$. As N goes to infinity, $\underline{\kappa}$ converges to $\frac{1}{7}\mu \approx .1\mu$, and $\overline{\kappa}$ converges to 4μ. The following proposition provides a complete characterization of equilibrium arbitrageur activity in the isosceles economy:

Proposition 9 (Equilibrium Networks) *Suppose the Isosceles Assumption holds. Then for any economy and any network architecture, there is a unique equilibrium network. The equilibrium distribution of arbitrageurs is a continuous function of* μ_{12}.

i. For the unrestricted network architecture, $N^{01} = N^{02} = \frac{1}{2}(N - N^{12})$, *with*

$$N^{12} = \begin{cases} 0 & \text{if } \mu_{12} \in [0, \underline{\kappa}] \\ \frac{N}{3} & \text{if } \mu_{12} = \mu \\ N & \text{if } \mu_{12} \in [\overline{\kappa}, 4\mu] \end{cases}$$

There is activity on all three links if and only if $\mu_{12} \in (\underline{\kappa}, \overline{\kappa})$, *and* N^{12} *is strictly increasing in* μ_{12} *on this interval.*

ii. For the h_0*-network architecture,* $N^{01} = N^{02} = \frac{N}{2}$*, for all* $\mu_{12} \in [0, 4\mu]$.

iii. For the h_1*-network architecture,* $N^{01} = N - N^{12}$*, with*

$$N^{12} = \begin{cases} \frac{1}{3}(N-2) & \text{if } \mu_{12} = 0 \\ \frac{N}{2} & \text{if } \mu_{12} = \mu \\ N & \text{if } \mu_{12} \in [\overline{\kappa}, 4\mu] \end{cases}$$

There is activity on both the admissible links $(1, 0)$ *and* $(1, 2)$ *if and only if* $\mu_{12} \in [0, \overline{\kappa})$*, and* N^{12} *is strictly increasing in* μ_{12} *on this interval.*

Consider $\mu_{12} = 0$. In this case exchanges 1 and 2 are identical to each other: $p^1 = p^2$. The equilibrium in the unrestricted network architecture entails no arbitrageur activity on link $(1, 2)$. For the h_1-network architecture, why don't we get the obvious $N^{12} = 0$ in light of the fact that exchanges 1 and 2 are identical and would not therefore provide for any arbitrage opportunities? The reason is of course that while 1 and 2 are identical in autarky, they are no longer identical in equilibrium. Given that some arbitrageurs arbitrage between 1 and 0, state prices on 1 in equilibrium no longer equal p^2, thereby inducing an arbitrage opportunity strong enough to support $\frac{N-2}{3} < \frac{N}{2}$ arbitrageurs.

It is also transparent that as $N \to \infty$, $\hat{p}^k(N) \to p^\lambda$, for any $k \in K$, in any of the network architectures studied in Proposition 9, as expected by Lemma 8. With the exception of the unrestricted and the h_1 networks when $\mu_{12} = 4\mu$, all networks satisfy $\mathcal{P}(\infty) = \{K\}$, so state prices converge to Walrasian state prices. The reason for this is that as $N \to \infty$, $\bar{\kappa}(N) \to 4\mu$, so for N large enough any $\mu_{12} < 4\mu$ will eventually satisfy $\mu_{12} < \bar{\kappa}$, in which case the networks eventually are connected networks that are maximal. The only exception is $\mu_{12} = 4\mu$ in which case $\mathcal{P}(\infty) = \{\{0\}, \{1, 2\}\}$, and yet state prices become Walrasian in the limit. The reason is that by the Isosceles assumption in that particular case[8] $p^0 = p^\lambda$. Now $\{p^0, \hat{p}^1(N), \hat{p}^2(N)\}$ will always lie on a straight line connecting p^1 to p^2. Since we know that $\hat{p}^1(N) - \hat{p}^2(N) \to 0$ by Lemma 4, that common limit is $p^0 = p^\lambda$ by symmetry. We see that $\mathcal{P}(\infty) = \{K\}$ is a sufficient but not a necessary condition for convergence to Walrasian pricing.

We define the symbol $>_n$ to mean "$>$ if $N \geq n$." Analogously, $>_\infty$ means "$>$ if N is sufficiently large" (recall that we always assume that $N \geq 3$).

Proposition 10 (Profit-maximizing network architectures) *Suppose the Isosceles Assumption holds. Then, arbitrageur profits in a unary network architecture are strictly lower than in any other network architecture, except if* $\mu_{12} \in [\overline{\kappa}, 4\mu]$ *in which case* $\Phi^{h_1} = \Phi = \Phi^{u_{12}}$*. For other network architectures, we have the following:*

[8] We know that $\mu_{12} = 4\mu_{01} = 4\mu_{02}$, so $2\mu_{01} + 2\mu_{02} = \mu_{12}$. It can then be checked that this equation can be rewritten as $\left(\omega^0 - \frac{\omega^1+\omega^2}{2}\right)^\top \Pi \left(\omega^0 - \frac{\omega^1+\omega^2}{2}\right) = 0$, i.e. $\omega^0 = \frac{\omega^1+\omega^2}{2} = \frac{(\omega^1+\omega^2)+\frac{1}{2}(\omega^1+\omega^2)}{3} = \frac{\omega^1+\omega^2+\omega^0}{3}$. By definition $p^\lambda = \mathbf{1} - \beta\frac{\omega^0+\omega^1+\omega^2}{3} = \mathbf{1} - \beta\omega^0 = p^0$.

i. For $\mu_{12} = 0$,

$$\Phi^{h_1} >_5 \Phi^{h_0} = \Phi$$

If $N = 4$, $\Phi^{h_1} = \Phi^{h_0}$, *and if* $N = 3$, $\Phi^{h_0} > \Phi^{h_1}$.

ii. For $\mu_{12} \in (0, \underline{\kappa}]$,

$$\Phi^{h_1} >_\infty \Phi^{h_0} = \Phi$$

iii. For $\mu_{12} \in (\underline{\kappa}, \mu)$,

$$\Phi^{h_1} >_\infty \Phi^{h_0} >_\infty \Phi$$

iv. For $\mu_{12} = \mu$,

$$\Phi^{h_0} = \Phi^{h_1} >_5 \Phi$$

If $N \leq 4$, $\Phi^{h_0} = \Phi^{h_1} < \Phi$

v. For $\mu_{12} \in (\mu, \overline{\kappa})$,

$$\Phi^{h_0} >_\infty \Phi^{h_1} >_\infty \Phi$$

vi. For $\mu_{12} \in [\overline{\kappa}, 4\mu]$,

$$\Phi^{h_0} > \Phi^{h_1} = \Phi$$

Note that, due to symmetry, we always have $\Phi^{h_2} = \Phi^{h_1}$, and also $\Phi^{u_{02}} = \Phi^{u_{01}}$.

The exercise of keeping p^0 and p^1 fixed and rotating p^2 covers all possible network shapes with three exchanges in spaces $\mathbb{R}^S$, $S \geq 2$. The parameter $\mu_{12} \in [0, 4\mu]$ measures, over and above the gains from trade between exchanges 1 and 2, the extent of direct competition between arbitrageurs due to the network structure, for given N. A higher level of μ_{12} means more product differentiation. $\mu_{12} = 0$ corresponds to extreme competition since the products traded with exchange 0 are identical in autarky, $\mu_{12} = \mu$ is the equilateral case, while $\mu_{12} = 4\mu$ corresponds to minimal competition since all three autarky state prices lie evenly spaced on a line and are therefore maximally differentiated.

It may be worthwhile to emphasize some of the results in this proposition:

Corollary 2 *Suppose the Isosceles Assumption holds. Then, a profit-maximizing network architecture is always hub-spoke (star), at least for large enough* N*. The unconstrained network architecture in equilibrium never leads to an optimal hub-spoke network (but may lead to a suboptimal hub-spoke network), at least for* N *large enough. Unary network architectures lead to the (weakly) lowest equilibrium arbitrage profits.*

For $\mu_{12} = \mu$, it doesn't matter which exchange is chosen to be the hub, and all hub-spoke networks are optimal for arbitrageurs. For $\mu_{12} > \mu$, the profit-maximizing hub is the central exchange, while for $\mu_{12} < \mu$ the profit-maximizing hub is an extreme exchange. Even the suboptimal hub is always at least as good as the unrestricted network. The restrictions implicit in certain central hub-spoke network configurations coordinate arbitrageur actions by pooling liquidity and by preventing prisoner's dilemma type deviations. The state prices of the optimal hub are in equilibrium (no necessarily in autarky) to some extent "in between" the two remaining state prices, therefore acting as a liquidity pool. Restricting networks further (in this three exchange scenario to unary ones) is counterproductive and always hurts equilibrium arbitrage profits.

We now provide some more detailed analysis of the various configurations.

$\underline{\mu_{12} = 0}$. When $\mu_{12} = 0$, the equilibrium unrestricted network entails no arbitrageur activity on link $(1, 2)$, i.e. the unrestricted network and the h_0-network have the same equilibrium distribution of arbitrageurs. Hence $\Phi = \Phi^{h_0}$. The h_0-network is optimal for arbitrageurs for $N = 3$. For $N = 3$, competition is weak, and with 0 as the hub, arbitrageurs (who are equally divided across the two active links) can exploit the larger initial mispricings across $(0, 1)$ and $(0, 2)$. In doing so, they pull the state prices toward the middle. However, for $N \geq 4$ the profit-maximizing network is the h_1-network. For $N \geq 4$, competition between the two links is direct (both links compete for resources on 0) and dissipates profits quickly even though autarky gains from trade are large. In this case, an extreme hub dominates, by providing respite from competition and providing arbitrage liquidity due to the increased product differentiation. Notice that the state prices of the extreme hub 1 in equilibrium lie halfway between $\hat{p}^0$ and $\hat{p}^2$ in order to equalize profits, so that in particular N^{01}/N^{12} is large and converges downwards to 2 as $N \to \infty$. The products are complementary in the sense that the arbitrageurs on $(1, 2)$ provide liquidity to the ones on $(0, 1)$ and vice versa, which is sufficient to compensate for the smaller gains from trade in autarky.

There is no arbitrage in autarky between 1 and 2. While it is true that competition between arbitrageurs pushes prices towards Walrasian ones, in the h_1 architecture arbitrageurs first create an arbitrage between 1 and 2 for N small, before larger competition eliminates said mispricing in turn.

The h_1 network is not an equilibrium when the architecture is unrestricted. Indeed, assume now that at the equilibrium of the h_1 architecture one arbitrageur is authorized to arbitrage $(0, 2)$. Clearly, since all depths are identical and since the distance between equilibrium state prices on 0 and 2 is larger than between any other pair (in fact it is twice the length of any other link), it is worthwhile for him to deviate. The reason for this is that the liquidity provision and the resulting externality are not fully internalized and lead to Prisoner's Dilemma deviations, yielding a suboptimal arrangement. The optimal network cannot be implemented as an equilibrium network short of imposing a ban on arbitrage between 0 and 2.

$\underline{\mu_{12} = \mu}$. Rotating p^2 towards p^1, one obvious benchmark is when the triangle is equilateral, with $\mu_{12} = \mu$. In this case all links are symmetric and attract the same number of arbitrageurs. As for hub-spoke networks, arbitrageurs are indifferent as to which one of the exchanges is the hub, due to symmetry. This economy will serve as the critical level of competition above which an extreme hub is preferred and below which the central hub provides higher profits. The unrestricted network is fully connected and profits are less than if any one link had been disallowed, say link $(0, 2)$. The reason is again similar to the one given for the case $\mu_{12} = 0$. If $(0, 2)$ is left unarbitraged, there is less pressure for the prices on 0 and 2 to converge, leaving the two arbitrages $(1, 0)$ and $(1, 2)$ more differentiated, and therefore more profitable, with arbitrageurs providing liquidity to each other to some extent (but less than in the previous case). As N increases, equilibrium state prices on 0 and 2 converge to p^λ along a parabola. It is therefore not hard to see that the distance between equilibrium state prices on 0 and 2 is high, so that if trade $(0, 2)$ was now allowed, arbitrageurs would deviate to the link $(0, 2)$. Again a Prisoner's Dilemma result obtains.

$\mu_{12} = 4\mu$. For $\mu_{12} \in [\overline{\kappa}, 4\mu]$, both the unrestricted network and the h_1-network have the same equilibrium distribution of arbitrageurs as the u_{12}-network, and therefore the same profits. Thus we have

$$\Phi^{h_0} > \Phi^{h_1} = \Phi = \Phi^{u_{12}} > \Phi^{u_{01}}.$$

The optimal network for arbitrageurs is the h_0-network. In equilibrium, the state price constellation is similar to the one when $\mu_{12} = 0$. In this case the equilibrium state price vector of exchange 0 lies in the middle and acts as a liquidity repository. Say, arbitraging activities involve moving consumption in state s from exchange 0, the central exchange, to exchange 2. This hub structure is an intelligent one, for by the assumption of all three state prices lying on a line, the arbitrageurs between 0 and 1 are laying off that consumption onto exchange 0, and there is therefore more to take away from 0 towards 2, at a lesser strain on prices. In that sense, the two different families of arbitrageurs complement each other by making the central exchange more liquid than it would otherwise be. In this instance, the competition effect we have just underscored, namely the network-induced lower degree of competition, is sufficient to compensate for the fact that the autarky gains from trade between the center and any one of the extremes is considerably less than the gains from trade between the two extremes, ignoring the middle exchange altogether. Indeed, since the layoff of consumption to the hub on the middle exchange is exactly the consumption taken away again by the other family of arbitrageurs, the net trade of both families together is in fact the trade of a single family if an extreme exchange is taken as the hub. This is identical to what we have seen in the $\mu_{12} = 0$ case.

8 Networks and Social Welfare

*** NOTE: This section is work in progress. ***

Associated with a network with asset structure $\{R^k\}$, there is a unique CWE with the corresponding equilibrium payoffs for each arbitrageur and investor. Equilibrium arbitrageur profits are given by (9). We now turn to the equilibrium utilities of investors. Using investor (k,i)'s first order condition, we can write his utility (1) as:

$$U^{k,i} = \omega_0^{k,i} + \mathbf{1}^\top \Pi \omega^{k,i} - \frac{\beta^{k,i}}{2} \omega^{k,i\top} \Pi \omega^{k,i} + \frac{\beta^{k,i}}{2} \|R^k \theta^{k,i}\|_2^2.$$

Note that $U^{k,i}$ depends on the asset structure only through the term $W^{k,i} := \beta^{k,i} \|R^k \theta^{k,i}\|_2^2$. We will find it convenient to refer to $W^{k,i}$ as the equilibrium utility of agent (k,i). From (2), we see that

$$R^k \theta^{k,i} = \frac{1}{\beta^{k,i}} P^k (p^{k,i} - \hat{p}^k), \qquad k \in K,$$

where $P^k := R^k (R^{k\top} \Pi R^k)^{-1} R^{k\top} \Pi$, the orthogonal projection in $L^2(\Pi)$ from $\mathbb{R}^S$ onto the asset span $\langle R^k \rangle$. Hence, we have

Lemma 9 (Equilibrium utilities) *Given a network with asset structure $\{R^k\}$, the equilibrium utility of investor (k,i), $k \in K$, is*

$$W^{k,i} = \frac{1}{\beta^{k,i}} \|P^k (p^{k,i} - \hat{p}^k)\|_2^2.$$

A network is optimal for an investor if it results in the highest equilibrium utility for the investor among all possible networks. A network is Pareto optimal for a group of agents if there is no alternative network that Pareto dominates it in equilibrium for this group. A network is socially optimal if it is Pareto optimal for the set of all agents, arbitrageurs and investors.

We say that investors on exchange k are homogeneous if they have the same no-trade valuations, i.e. $p^{k,i} = p^k$, for all $i \in I^k$. We refer to an economy in which investors are homogeneous within each exchange as a clientèle economy. From the point of view of arbitrageurs, each clientèle $k \in K$ consists of agents with identical characteristics.

We will focus now on a clientèle economy. Lemma 9 gives us the following welfare index for clientèle $k \in K$:

$$W^k := \sum_{i \in I^k} W^{k,i} = \frac{1}{\beta^k} \|P^k (p^k - \hat{p}^k)\|_2^2.$$

We can think of W^k, for an arbitrary asset structure $\{R^k\}$, as the inter-exchange gains from trade reaped by exchange k in moving from autarky to the arbitraged equilibrium. From (4), $p^k - \hat{p}^k = \beta^k R^k y^k$, which is in the span of R^k (regardless of whether **S** holds or not). Therefore,

$$W^k = \frac{1}{\beta^k} \|p^k - \hat{p}^k\|_2^2. \tag{17}$$

Note that $W^k = \beta^k \|R^k y^k\|_2^2$, so that the gains from trade are proportional to the magnitude of state-contingent consumption trading volume. In particular this implies that for a given architecture with all cells of $\mathcal{P}_0$ containing only two elements, the outcome of the game is Pareto optimal in view of the fact that $R^k y^k = -R^\ell y^\ell$. All the subtleties arise from the network externalities.

Consider first the class of unary networks. The next proposition asserts that the profit-maximizing unary network architecture (given in Proposition 6) is in fact socially optimal.

Proposition 11 *In a clientèle economy, the profit-maximizing unary network architecture $u_{k^*\ell^*}$ maximizes the egalitarian social welfare function $\sum_{k \in K} W^k$ in the class of unary networks. It is therefore socially optimal in this class.*

We now turn to a welfare analysis of networks in a clientèle economy more generally, but imposing the Isosceles Assumption. The welfare of investors on exchange k is given by (17). Since the β's are the same for all exchanges, we take them all to be one, without loss of generality, so that $W^k = \|p^k - \hat{p}^k\|_2^2$.

Lemma 10 (Utilities: Isosceles case) *Suppose the Isosceles Assumption holds. Then, for a given network, welfare on exchange k is:*

$$\begin{aligned} W^k = &\frac{1}{\left[4(N+1)+3\overline{N}\right]^2} \cdot \\ &\Big[2\left(\overline{N}+2N^{k\ell}\right)\left(\overline{N}+N^{k\ell}+N^{k\ell'}\right)\|p^k-p^\ell\|_2^2 \\ &\quad + 2\left(\overline{N}+2N^{k\ell'}\right)\left(\overline{N}+N^{k\ell}+N^{k\ell'}\right)\|p^k-p^{\ell'}\|_2^2 \\ &\quad - \left(\overline{N}+2N^{k\ell}\right)\left(\overline{N}+2N^{k\ell'}\right)\|p^\ell-p^{\ell'}\|_2^2\Big], \end{aligned}$$

where

$$\overline{N} := N^{k\ell}N^{k\ell'} + N^{k\ell}N^{\ell\ell'} + N^{k\ell'}N^{\ell\ell'},$$

and $\ell \neq \ell'$ distinct from k.

In order to distinguish welfare across networks, we will reserve the notation W^k for welfare on exchange k in the unrestricted network, and denote welfare on exchange k in the h_ℓ and $u_{k\ell}$-networks by W^{k,h_ℓ} and $W^{k,u_{\ell\ell'}}$ respectively, all for given $\{N^{k\ell}\}$.

For an equilibrium network, with endogenous distribution of arbitrageurs, we denote welfare by $\mathcal{W}$, with the appropriate superscript. From Lemma 10, we have

$$W^0 = \left[\frac{x(2N-3x+2)}{4(N+1)+3x(2N-3x)}\right]^2 \cdot (4\mu - \mu_{12}), \tag{18}$$

$$W^1 = W^2 = \frac{1}{[4(N+1)+3x(2N-3x)]^2} \cdot \Big[x^2(2N-3x+2)^2\mu + 2\big[x(2N-3x-4)+2N\big] \cdot \big[x(2N-3x-1)+N\big]\mu_{12}\Big],$$

where x solves the quadratic (29). In the h_0-network, assuming that there is some arbitrageur activity on links $(0,1)$ and $(0,2)$, we have $x = \frac{N}{2}$, so that

$$\mathcal{W}^{0,h_0} = \left[\frac{N}{3N+4}\right]^2 \cdot (4\mu - \mu_{12}),$$

$$\mathcal{W}^{1,h_0} = \mathcal{W}^{2,h_0} = \left[\frac{N}{(N+4)(3N+4)}\right]^2 \cdot \Big[(N+4)^2\mu + 2N(N+2)\mu_{12}\Big].$$

Also from Lemma 10, noting that $N^{02} = 0$ in the h_1-network:

$$W^{0,h_1} = W^{0,h_2} = \left[\frac{x}{4(N+1)+3x(N-x)}\right]^2 \cdot \Big[4(N-x+1)^2\mu - (N-x)(N-x+2)\mu_{12}\Big], \tag{19}$$

$$W^{1,h_1} = W^{2,h_2} = \frac{1}{[4(N+1)+3x(N-x)]^2} \cdot \Big[x^2(N-x+2)^2\mu + 2(N-x)(x+2)[N+x(N-x)]\mu_{12}\Big],$$

$$W^{1,h_2} = W^{2,h_1} = \left[\frac{N-x}{4(N+1)+3x(N-x)}\right]^2 \cdot \Big[x^2\mu + 2(x+2)(x+1)\mu_{12}\Big],$$

where x solves the quadratic (33). Finally, for minimally connected networks, $\mathcal{W}^{k,u_{\ell\ell'}} = 0$ if both ℓ and ℓ' are distinct from k. Otherwise,

$$\mathcal{W}^{0,u_{01}} = \mathcal{W}^{0,u_{02}} = \mathcal{W}^{1,u_{01}} = \mathcal{W}^{2,u_{02}} = \left[\frac{N}{2(N+1)}\right]^2 \mu,$$

$$\mathcal{W}^{1,u_{12}} = \mathcal{W}^{2,u_{12}} = \left[\frac{N}{2(N+1)}\right]^2 \mu_{12}.$$

Proposition 12 (Investor Welfare) *Suppose the Isosceles Assumption holds. In the benchmark cases, we have the following:*

i. For $\mu_{12} = 4\mu$, $\mathcal{W}^{0,u_{01}} = \mathcal{W}^{0,u_{02}} > \mathcal{W}^{0,h_0} = \mathcal{W}^0 = \mathcal{W}^{0,h_1}$

ii. For $\mu_{12} = \mu$, $\mathcal{W}^{0,h_0} > \mathcal{W}^{0,u_{01}} = \mathcal{W}^{0,u_{02}} > \mathcal{W}^0 > \mathcal{W}^{0,h_1}$

iii. For $\mu_{12} = 0$,[9] $\mathcal{W}^{0,h_0} = \mathcal{W}^0 > \mathcal{W}^{0,h_1} \geq \mathcal{W}^{0,u_{01}} = \mathcal{W}^{0,u_{02}}$

and

i. For $\mu_{12} = 4\mu$, $\mathcal{W}^{1,u_{12}} = \mathcal{W}^{1,h_1} = \mathcal{W}^1 = \mathcal{W}^{1,h_2} > \mathcal{W}^{1,h_0} > \mathcal{W}^{1,u_{01}}$

ii. For $\mu_{12} = \mu$, $\mathcal{W}^{1,h_1} > \mathcal{W}^{1,u_{01}} = \mathcal{W}^{1,u_{12}} > \mathcal{W}^1 > \mathcal{W}^{1,h_0} = \mathcal{W}^{1,h_2}$

iii. For $\mu_{12} = 0$, $\mathcal{W}^{1,u_{01}} \geq \mathcal{W}^{1,h_1} > \mathcal{W}^1 = \mathcal{W}^{1,h_0} > \mathcal{W}^{1,h_2} > \mathcal{W}^{1,u_{12}}$

9 Conclusion

*** To be written ***

[9]Recall that, in the h_1-network for the case of $\mu_{12} = 0$, if $N = 3$, all arbitrageur activity is concentrated on link $(0, 1)$. Hence $\mathcal{W}^{0,h_1} = \mathcal{W}^{0,u_{01}}$. If $N \geq 4$, $\mathcal{W}^{0,h_1} > \mathcal{W}^{0,u_{01}}$. The same consideration applies to welfare on exchange 1.

A Appendix

Proof of Lemma 1 Using (3), we can write the Lagrangian for arbitrageur $n \in N^{k\ell}$ as follows:

$$\mathcal{L} = \sum_{m\in\{k,\ell\}} [p^m - \beta^m R^m y^{m,n}_{k\ell} - \beta^m R^m y^{m,\backslash n}]^\top \Pi R^m y^{m,n}_{k\ell} - \psi^\top \Pi \sum_{m\in\{k,\ell\}} R^m y^{m,n}_{k\ell},$$

where ψ is the Lagrange multiplier vector attached to the no-default constraints, and can be interpreted as a (shadow) state price deflator of the arbitrageur. The first order conditions are:

$$R^{m\top}\Pi[p^m - \beta^m R^m y^{m,\backslash n} - 2\beta^m R^m y^{m,n}_{k\ell} - \psi] = 0, \qquad m \in \{k,\ell\}$$

together with complementary slackness:

$$\psi \geq 0, \quad \sum_{m\in\{k,\ell\}} R^m y^{m,n}_{k\ell} \leq 0, \quad \text{and} \quad \psi_s \cdot \left[\sum_{m\in\{k,\ell\}} R^m y^{m,n}_{k\ell}\right]_s = 0, \quad \forall s. \tag{20}$$

We can rewrite the first order conditions as follows:

$$R^{m\top}\Pi[\hat{p}^m - \beta^m R^m y^{m,n}_{k\ell} - \psi] = 0, \qquad m \in \{k,\ell\}. \tag{21}$$

It is easy to check that a solution to (20) and (21) is given by (5), with

$$\psi = \left(\frac{1}{\beta^k} + \frac{1}{\beta^\ell}\right)^{-1} \left(\frac{1}{\beta^k}\hat{p}^k + \frac{1}{\beta^\ell}\hat{p}^\ell\right). \tag{22}$$

The supplies given by (5) are feasible since $\hat{p}^k - \hat{p}^\ell$ is in the span of both R^k and R^ℓ by **S**. The Lagrange multiplier vector ψ is nonnegative if $\hat{p}^k$ and $\hat{p}^\ell$ are both nonnegative. This is indeed the case, as we will verify later (Lemma 3). The no-default constraints hold with equality. This argmax is in fact unique since the program is globally concave. Thus the CWE is symmetric, i.e. $y^{m,n}_{k\ell}$ does not depend on n. ■

Proof of Lemma 3 It is easy to see that M is dominant diagonal in the sense of Hadamard, since, with j in the kth block and with each block itself diagonal, $|M_{jj}| > \sum_{i=0}^{(K+1)S} |M_{ji}|$ iff $|1 + \beta^k\alpha^k| > \sum_{\ell\neq k,\ell=0}^{K+1} |-\beta^k\alpha^{k\ell}| = \sum_{\ell\neq k}\beta^k\alpha^{k\ell} = \beta^k\alpha^k$ iff $1 > 0$. It is well known (see for instance Takayama (1985) Theorem 4.C.1) that a dominant diagonal matrix is non-singular, and so the solution is unique. But we can also establish nonnegativity. Since $M_{ii} > 0$ for all $i = 0, \ldots, (K+1)S$, and $M_{ij} \leq 0$ for all $i \neq j$, by Theorem 4.C.3 in Takayama (1985) there exists a unique $\hat{p} \geq 0$ such that $M\hat{p} = p$ for every $p \geq 0$ iff M is dominant diagonal. Finally, $M^{-1} \geq 0$ by Theorem 4.C.4 in Takayama (1985). ■

Proof of Lemma 4 Consider the equation determining equilibrium state prices (7), and let $N \to \infty$. What matters for connectedness in the limit is $\rho^{k\ell} :=$

$\lim_{N\to\infty} \frac{\alpha^{k\ell}}{\alpha^k}(N)$. In matrix form, $[Z \otimes I_{S\times S}]\hat{p}(\infty) = \hat{p}(\infty)$, with $Z_{kk} = 0$ and $Z_{k\ell} = \rho^{k\ell} \geq 0$, $k, \ell \in G$. By Theorem 8.1. in Nikaido (1968), Z is indecomposable by the assumption that $G \in \mathcal{P}(\infty)$. It is also clear that 1 is an eigenvalue of Z with eigenvector $\mathbf{1} := \{1, \dots, 1\}$. Denote by $\lambda(Z)$ the Frobenius-Perron root of Z. By Theorem 7.4 in Nikaido (1968), since Z is indecomposable it must be that $\lambda(Z)$ is a simple root of the characteristic equation. Since Z is indecomposable, from Theorem 7.5 in Nikaido (1968) which states that $\lambda(Z)$ lies in between the minimal row sum and the maximal row sum, we can deduce that $\lambda(Z) = 1$. By Theorem 7.3 in Nikaido (1968), the eigenvector associated with $\lambda(Z) = 1$ is unique (up to multiplication by scalars), so that $\mathbf{1}$ is the only eigenvector (up to multiplication by scalars) associated with the eigenvalue 1. So fix a state $s \in S$ and define $\hat{p}_s(\infty) := \{\hat{p}^k_s(\infty)\}_{k\in G}$, an element of $\mathbb{R}^{|G|}$. The system $[Z \otimes I_{S\times S}]\hat{p}(\infty) = \hat{p}(\infty)$ can be reexpressed state by state as $Z\hat{p}_s(\infty) = \hat{p}_s(\infty)$, $s \in S$. By the above, $\hat{p}_s(\infty) = c_s\mathbf{1}$ for some scalar c_s, i.e. $\hat{p}^k_s(\infty) = \hat{p}^\ell_s(\infty) = c_s$, all k, ℓ in G. Since $s \in S$ was arbitrary, we must have $\hat{p}^k(\infty) = \hat{p}^\ell(\infty)$, all k, ℓ in G.

We now show that state prices converge to Walrasian ones. Define $\bar{N}$ large enough so that for all $N > \bar{N}$, $\mathcal{P}(N) = \mathcal{P}(\infty)$. By the above, $\hat{p}^k(\infty) = \bar{p}_G$ for some $\bar{p}_G$, all $k \in G$. Now multiply equation (7) by λ^k_G and sum over $k \in G$ to get

$$\sum_{k\in G} \lambda^k_G \hat{p}^k(N) = p^\lambda_G \text{ for any } N > \bar{N}$$

and in particular also for $N \to \infty$. Thus $p^\lambda_G = \sum_{k\in G} \lambda^k_G \hat{p}^k(\infty) = \sum_{k\in G} \lambda^k_G \bar{p}_G = \bar{p}_G$. ■

Proof of Lemma 5 Assuming **S**, it is straightforward to solve for $\{\hat{p}^k\}_{k\in K}$ using Lemma 2. Now, we verify that **S** and $\mathbf{S^{ho}}$ are equivalent. Under **S**, it is apparent from the solution to $\{\hat{p}^k\}_{k\in K}$ that we have just derived that $\hat{p}^k - \hat{p}^0 = (1+\beta^k\alpha^{0k})^{-1}(p^k - p^\gamma)$. Hence **S** implies $\mathbf{S^{ho}}$. Conversely, under $\mathbf{S^{ho}}$, we can solve for $\{\hat{p}^k\}_{k\in K}$ assuming that the arbitrageur's no-default constraints hold with equality, getting the same solution as under **S**. Then, $\hat{p}^k \geq 0$, for all k, due to the standing assumption that $p^k \geq 0$, for all k. Also, $\hat{p}^k - \hat{p}^0$, being collinear with $p^k - p^\gamma$, lies in $\langle R^k\rangle \cap \langle R^0\rangle$, for all $k \neq 0$. Thus **S** is satisfied. This also justifies the assumption that the no-default constraints hold with equality (as in the proof of Lemma 1). ■

Proof of Proposition 1 It is clear that any asset structure satisfying **S** is a Nash equilibrium of the game without exclusivity by Corollary 1. Within the class of asset structures satisfying **S**, $\{\hat{p}^k\}$ is the same. The equilibrium supplies of state-contingent consumption, and as a result profits, are the same for any such asset structure. So adding further assets on a non-exclusive basis leads to a new equilibrium which coincides with the old one. There is therefore no incentive to deviate.

Next we show the stronger result that any asset structure satisfying **S** is a Nash equilibrium in the game where innovation can be exclusive. We wish to show that the equilibrium of the trading subgame is unaffected by the innovation. So assume that trades by all other arbitrageurs, $y^{\setminus n}$ are unaffected. Suppose arbitrageur $n \in N^{k,\ell}$ deviates at the design stage of the game by introducing additional assets with payoff matrix D^k on exchange k and D^ℓ on exchange ℓ. Let $y_D^{m,n}$ be arbitrageur n's supply of the additional assets on exchange $m \in \{k, \ell\}$. Note that we allow the deviating arbitrageur to have monopolistic access to these assets, so that he is the sole supplier. Using (3), we can write the Lagrangian for the optimization problem of the arbitrageur as

$$\mathcal{L} = \sum_{m\in\{k,\ell\}} \left[p^m - \beta^m(R^m y_{k\ell}^{m,n} + R^m y^{m,\setminus n} + D^m y_D^{m,n})\right]^\top \Pi \left[R^m y_{k\ell}^{m,n} + D^m y_D^{m,n}\right]$$
$$- \psi^\top \Pi \sum_{m\in\{k,\ell\}} (R^m y_{k\ell}^{m,n} + D^m y_D^{m,n}),$$

where ψ is the Lagrange multiplier vector on the no-default constraints. The first order conditions with respect to $y^{m,n}$ and $y_D^{m,n}$ are $R^{m\top}\Pi Y^m = 0$, and $D^{k\top}\Pi Y^m = 0$, respectively, where

$$Y^m := p^m - \beta^m R^m y^{m,\setminus n} - 2\beta^m R^m y_{k\ell}^{m,n} - 2\beta^m D^m y_D^{m,n} - \psi.$$

It is easy to verify that a solution to the problem is $y_D^{m,n} = 0$, $y_{k\ell}^{m,n}$ given by (5), and ψ given by (22). Indeed, at these values, Y^k and Y^ℓ are both zero. The supplies given by (5) are feasible since $\hat{p}^k - \hat{p}^\ell$ is in the span of both R^k and R^ℓ. The Lagrange multiplier vector ψ is nonnegative since $\hat{p}^k$ and $\hat{p}^\ell$ that are both nonnegative. The no-default constraints hold with equality. This solution is in fact unique since the program is globally concave. Thus the deviating arbitrageur has nothing to gain by deviating. state prices are unaffected, justifying the assumption that $y^{\setminus n}$ is constant.

■

Proof of Lemma 7 Using Lemma 1, the equilibrium profit of arbitrageur $n \in N^{k\ell}$ is

$$\begin{aligned}\varphi^{k\ell} &= (q^k - q^\ell)^\top y_{k\ell}^{k,n} \\ &= (\hat{p}^k - \hat{p}^\ell)^\top \Pi R^k y_{k\ell}^{k,n} \\ &= \frac{1}{\beta^k + \beta^\ell} \cdot \|\hat{p}^k - \hat{p}^\ell\|_2^2.\end{aligned}$$

Profits in the h_0-network are now easily calculated, using (8). For the case of three exchanges, we have from Lemma 6:

$$\hat{p}^k - \hat{p}^\ell = \frac{1}{1+N+\nu} \cdot \left[(1+N-N^{k\ell}-\beta^k\alpha^{k\ell'})(p^k - p^\ell) - \beta^k\alpha^{k\ell'}(p^\ell - p^{\ell'}) - \beta^\ell\alpha^{\ell\ell'}(p^{\ell'} - p^\ell)\right].$$

Substituting in (9),

$$\varphi^{k\ell} = \frac{1}{(1+N+\nu)^2(\beta^k+\beta^\ell)} \cdot \\ \Big[(1+N-N^{k\ell}-\beta^k\alpha^{k\ell'})^2 \, \|p^k - p^\ell\|_2^2 \\ + (\beta^\ell\alpha^{\ell\ell'} - \beta^k\alpha^{k\ell'})^2 \, \|p^\ell - p^{\ell'}\|_2^2 \\ + 2(1+N-N^{k\ell}-\beta^k\alpha^{k\ell'})(\beta^\ell\alpha^{\ell\ell'} - \beta^k\alpha^{k\ell'})(p^k-p^\ell)^\top\Pi(p^\ell - p^{\ell'})\Big].$$

Now

$$\|p^k - p^{\ell'}\|_2^2 = \|p^k - p^\ell + p^\ell - p^{\ell'}\|_2^2 \\ = \|p^k - p^\ell\|_2^2 + \|p^\ell - p^{\ell'}\|_2^2 + 2(p^k - p^\ell)^\top\Pi(p^\ell - p^{\ell'}),$$

so that

$$2(p^k - p^\ell)^\top\Pi(p^\ell - p^{\ell'}) = \|p^k - p^{\ell'}\|_2^2 - \|p^k - p^\ell\|_2^2 - \|p^\ell - p^{\ell'}\|_2^2.$$

Substituting this in the above expression for $\varphi^{k\ell}$ we get the required result. Finally, profits in the $u_{k\ell}$-network are obtained by specializing the three-exchange case two exchanges, k and ℓ. ∎

Proof of Lemma 8 Any $G_0 \in \mathcal{P}_0$ can be written as $G_0 = \cup_{i\in I} G_i$, $G_i \in \mathcal{P}(\infty)$. By Lemma 4, for each $k \in G_i$, $\hat{p}^k(N) \to p^\lambda_{G_i}$. Assume now that $p^\lambda_{G_i} \neq p^\lambda_{G_j}$, $i \neq j$, $i, j \in I$. Pick $k \in G_i$ and $\ell \in G_j$. Since $G_i \subset G_0$ and $G_j \subset G_0$, $(k,\ell) \in \mathcal{A}$. Profits for an arbitrageur across (k,ℓ) are $\varphi^{k,\ell}(\infty) = \frac{1}{\beta^k+\beta^\ell}\|\hat{p}^k(\infty) - \hat{p}^\ell(\infty)\|_2^2 = \frac{1}{\beta^k+\beta^\ell}\|p^\lambda_{G_i} - p^\lambda_{G_j}\|_2^2 > 0$. By the assumption of an equilibrium in the network formation game, there must not be any profitable deviations. Now for $N > 0$, there must be arbitrageurs arbitraging within some subset of $\mathcal{P}(\infty)$, so we may assume without loss of generality that it is within G_i. Since by Lemma 4 profits converge to zero for all arbitrageurs arbitraging n, m in G_i, we have a contradiction in that $\varphi^{k\ell} > \varphi^{nm}$. We therefore also need that $\varphi^{k\ell}(N) \to 0$, i.e. that $p^\lambda_{G_i} = p^\lambda_{G_j}$ for all $i, j \in I$. It follows that there is one state price on all of G_0 in the limit, $p^\lambda_{G_0} = p^\lambda_G$, all $G \in \mathcal{P}(\infty)$. ∎

Proof of Proposition 8 Let $x = N^{01}$ and $y = N^{02}$, and define $b_k := \frac{\beta^0}{\beta^0+\beta^k}$, for $k = 1, 2$. Using (10) or (11), we find that

$$\varphi^{01}(x,y) = \frac{[1+(1-b_2)y][(1+y)\mu_{01} - b_1 y\mu_{02}] + y(1+y)(b_1+b_2-2b_1b_2)\mu_{12}}{[1+N+xy(1-b_1b_2)]^2},$$

$$\varphi^{02}(x,y) = \frac{[1+(1-b_1)x][(1+x)\mu_{02} - b_2 x\mu_{01}] + x(1+x)(b_1+b_2-2b_1b_2)\mu_{12}}{[1+N+xy(1-b_1b_2)]^2}.$$

The equilibrium distribution of arbitrageurs (x, y) is given (ignoring integer constraints) by the solution to $\varphi^{01}(x,y) = \varphi^{02}(x,y)$, with $x + y = N$, a quadratic equation, provided a real solution exists and is an element of $[0, N]$:

$$x^2(1-b_1b_2)(\mu_{02}-\mu_{01}) + (2x-N)H + (N+1)(\mu_{02}-\mu_{01}) = 0, \tag{23}$$

where

$$H := (N+1)(1-b_2)\mu_{01} + [1-b_1-Nb_1(1-b_2)]\mu_{02} + (N+1)(b_1+b_2-2b_1b_2)\mu_{12}.$$

It can be verified that $H > 0$ unless $\mu_{01} = \mu_{02} = \mu_{12} = 0$, in which case $H = 0$. The result follows. ■

Proof of Proposition 9 We begin by deriving expressions for equilibrium arbitrageur profits for the different networks. For ease of notation we let $x = N^{01}, y = N^{02}$, and $z = N^{12}$. From (11), we have

$$\varphi^{01}(x,y,z) = \frac{(y+2z+2)^2\mu + (y-z)(2y+z+2)\mu_{12}}{4(N+\nu+1)^2}, \tag{24}$$

$$\varphi^{02}(x,y,z) = \frac{(x+2z+2)^2\mu + (x-z)(2x+z+2)\mu_{12}}{4(N+\nu+1)^2}, \tag{25}$$

$$\varphi^{12}(x,y,z) = \frac{(x-y)^2\mu + (x+2y+2)(2x+y+2)\mu_{12}}{4(N+\nu+1)^2}. \tag{26}$$

After some straightforward algebra, we see that $\varphi^{01} = \varphi^{02}$ if and only if

$$(x-y)\big[(x+y+4)\mu + 2(x+y+1)\mu_{12} + z(4\mu-\mu_{12})\big] = 0,$$

i.e. if and only if $x = y$. Using this fact, we can now write:

$$\varphi^{01}(x,x,N-2x) = 4\cdot\frac{(2N-3x+2)^2\mu + (3x-N)(N+2)\mu_{12}}{[4(N+1)+3x(2N-3x)]^2}, \tag{27}$$

$$\varphi^{12}(x,x,N-2x) = \frac{4(3x+2)^2\mu_{12}}{[4(N+1)+3x(2N-3x)]^2}. \tag{28}$$

The equilibrium x is determined by equating (27) and (28), provided there is activity on all three links (there is activity on $(0,1)$ if and only if there is activity on $(0,2)$). It solves the following quadratic equation:

$$9(\mu_{12}-\mu)x^2 + (3x-N)L + 4(N+1)(\mu_{12}-\mu) = 0, \tag{29}$$

where

$$L := 4(N+1)\mu - (N-2)\mu_{12}.$$

Note that L is always strictly positive.

We can specialize the above analysis of unrestricted networks to hub-spoke networks. For the h_0-network, we have $z \equiv 0$ and $x = y = \frac{N}{2}$. From (27),

$$\Phi^{h_0} = 16\cdot\frac{(N+4)^2\mu + 2N(N+2)\mu_{12}}{(N+4)^2(3N+4)^2}. \tag{30}$$

For the h_1-network, we need to go back to equations (24) and (26), which give us:

$$\varphi^{01}(x, 0, N-x) = 4 \cdot \frac{4(N-x+1)^2\mu - (N-x)(N-x+2)\mu_{12}}{[4(N+1)+3x(N-x)]^2}, \tag{31}$$

$$\varphi^{12}(x, 0, N-x) = 4 \cdot \frac{x^2\mu + 2(x+2)(x+1)\mu_{12}}{[4(N+1)+3x(N-x)]^2}. \tag{32}$$

The equilibrium x is obtained by equating (31) and (32), provided there is activity on both links $(0,1)$ and $(1,2)$. It solves the following quadratic equation:

$$3(\mu_{12} - \mu)x^2 + (2x - N)L + 4(N+1)(\mu_{12} - \mu) = 0. \tag{33}$$

We can now analyze the equilibrium of the network game. Consider first the unrestricted network. For $\mu_{12} = \mu$, we have $x = \frac{N}{3}$, from (29). For $\mu_{12} \neq \mu$, the solutions to the quadratic (29) are:

$$x_\pm = \frac{3(N-2)\mu_{12} - 12(N+1)\mu \pm \sqrt{D}}{18(\mu_{12} - \mu)}, \tag{34}$$

where D is the discriminant:

$$D = 27(N+2)^2(4\mu - \mu_{12})\mu_{12}.$$

Clearly $D \geq 0$ for all $\mu_{12} \in [0, 4\mu]$, which implies that the roots x_+ and x_- are real.

Since $N^{01} = N^{02}$, it must be the case that $N^{01} \in [0, \frac{N}{2}]$. We first verify that $x_- \notin [0, \frac{N}{2}]$, and can therefore be discarded. In the formula (34) for x_-, the numerator is always negative. Hence, if $\mu_{12} > \mu$, then $x_- < 0$. For the case where $\mu_{12} < \mu$, we see from (34) that $x_- > \frac{N}{2}$ iff

$$\sqrt{D} > -3(N+4)\mu - 6(N+1)\mu_{12}$$

which is always satisfied, and so x_- can be discarded.

We now analyze the other root, x_+. In the formula (34) for x_+, the numerator is positive if and only if

$$Q := -(N^2 + 2N + 4)\mu_{12}^2 + (5N^2 + 10N + 8)\mu\mu_{12} - 4(N+1)^2\mu^2 > 0.$$

We can write Q as follows:

$$Q = (\mu_{12} - \mu)(\overline{\kappa} - \mu_{12})(N^2 + 2N + 4).$$

Hence $x_+ > 0$ if and only if $\mu_{12} < \overline{\kappa}$. At $\mu_{12} = \overline{\kappa}$, $x_+ = 0$.

Now we consider whether $x_+ < \frac{N}{2}$. For $\mu_{12} > \mu$, this inequality can be rewritten as

$$\sqrt{D} < 3(N+4)\mu + 6(N+1)\mu_{12}.$$

Squaring both sides and collecting terms, we get

$$(\mu_{12} - \mu)(\mu_{12} - \underline{\kappa}) > 0,$$

For $\mu_{12} < \mu$, the same analysis holds with the inequalities reversed. Therefore, $x_+ < \frac{N}{2}$ if and only if $\mu_{12} > \underline{\kappa}$. At $\mu_{12} = \underline{\kappa}$, $x_+ = \frac{N}{2}$.

Thus we have established that $x_+ \in [0, \frac{N}{2}]$ for $\mu_{12} \in [\underline{\kappa}, \overline{\kappa}]$. Hence $N^{01} = x_+$ in this interval. For $\mu_{12} \in (\overline{\kappa}, 4\mu]$, we have already shown that both x_+ and x_- are negative, i.e. both the zeros of the quadratic $g(x) := \varphi^{01}(x, x, N - 2x) - \varphi^{12}(x, x, N - 2x)$ are negative. Moreover, using the profit expressions (27) and (28), $g(0) < 0$. This shows that $g(x) < 0$ for all $x \geq 0$, so that arbitrageurs are always better off arbitraging the link between 1 and 2. Thus $N^{12} = N$.

For $\mu_{12} \in [0, \underline{\kappa})$, both the zeros of $g(x)$ are greater than $\frac{N}{2}$. Moreover, $g(\frac{N}{2}) > 0$, so it must be the case that $g(x) > 0$ for all $x \leq \frac{N}{2}$. It follows that $N^{12} = 0$.

Finally, we establish the monotonicity property of N^{12} in the interval $(\underline{\kappa}, \overline{\kappa})$. Since $N^{01} = x_+$ in this interval, it suffices to implicitly differentiate the quadratic (29) and show that $\frac{dx}{d\mu_{12}} < 0$. We assume for the moment that $\mu_{12} \neq \mu$. Then we can write (29) as follows:

$$9x^2 + (3x - N)L' + 4(N + 1) = 0$$

where $L' := L/(\mu_{12} - \mu)$. Implicitly differentiating this equation with respect to μ_{12}, we get

$$3[6x(\mu_{12} - \mu) + L]\frac{dx}{d\mu_{12}} = (N - 3x)(\mu_{12} - \mu)\frac{dL'}{d\mu_{12}}. \tag{35}$$

Now, $\frac{dL'}{d\mu_{12}} < 0$, and it is clear from an inspection of (29) that $N - 3x > 0$ if an only if $\mu_{12} - \mu > 0$. Hence the right hand side of (35) is negative, so that $\frac{dx}{d\mu_{12}} < 0$ if and only if $6x(\mu_{12} - \mu) + L > 0$. Since $L > 0$, we are done if $\mu_{12} > \mu$. If $\mu_{12} < \mu$, we have the following (note that, since $x = y$, the maximum value of x is $N/2$):

$$\begin{aligned} 6x(\mu_{12} - \mu) + L &\geq 3N(\mu_{12} - \mu) + L \\ &= (N + 4)\mu + 2(N + 1)\mu_{12} \\ &> 0. \end{aligned}$$

For $\mu_{12} = \mu$, we can establish the result by a similar argument, implicitly differentiating (29) with respect to μ_{12}.

This completes the proof for the unrestricted network. The equilibrium distribution of arbitrageurs for the h_0-network is immediate from Proposition 8. For the h_1-network, the relevant quadratic is (33), for which the discriminant D is:

$$D = -8(N + 1)(N + 4)\mu_{12}^2 + 4(7N + 10)(N + 4)\mu\mu_{12} + 16(N + 1)^2\mu^2.$$

We can think of D as a quadratic in μ_{12}. It is straightforward to check that $D > 0$ at $\mu_{12} = 0$ and at $\mu_{12} = 4\mu$, and that $\frac{\partial^2 D}{(\partial \mu_{12})^2} < 0$. Therefore, $D > 0$ for all $\mu_{12} \in [0, 4\mu]$, which implies that the roots x_+ and x_- of (33) are real.

The rest of the argument for the h_1-network is more or less identical to that for the unrestricted network, and is omitted. Note that in this case, we need to check that the relevant root lies in $[0, N]$, rather than $[0, \frac{N}{2}]$. It turns out that $x_+ < N$, with no restriction on μ_{12}. ■

Proof of Proposition 10 For the $u_{k\ell}$-network, profits are given by (12):

$$\Phi^{u_{k\ell}} = \frac{\mu_{k\ell}}{(N+1)^2}. \tag{36}$$

Using (30) and (36),

$$\Phi^{h_0} \geq 4 \cdot \frac{(N+4)^2\mu_{12} + 8N(N+2)\mu_{12}}{(N+4)^2(3N+4)^2} = \frac{4\mu_{12}}{(N+4)^2} > \Phi^{u_{12}},$$

where the last inequality is due to $N \geq 3$. Also,

$$\Phi^{h_0} \geq \frac{16\mu}{(3N+4)^2} > \Phi^{u_{01}}.$$

Therefore, the h_0-network strictly dominates any unary network.

We now turn to the equilibrium unrestricted network. From Proposition 9, $\Phi = \Phi^{h_0}$ for $\mu_{12} \in [0, \underline{\kappa}]$, and we have already shown that $\Phi^{h_0} > \Phi^{u_{k\ell}}$. Also, $\Phi = \Phi^{u_{12}} > \Phi^{u_{01}}$ for $\mu_{12} \in [\overline{\kappa}, 4\mu]$. For $\mu_{12} \in (\underline{\kappa}, \overline{\kappa})$, it easy to check that $\Phi > \Phi^{u_{12}}$ by comparing (28) and (36). Since $\Phi^{u_{12}} \geq \Phi^{u_{01}}$ for $\mu_{12} \geq \mu$, it remains to show that $\Phi > \Phi^{u_{01}}$ for $\mu_{12} < \mu$. From Proposition 9, $\frac{N}{2} \geq x > \frac{N}{3}$ in this case, so that, using (27) and (36):

$$\Phi \geq \left[\frac{2(2N-3x+2)}{4(N+1)+3x(2N-3x)}\right]^2 \cdot \mu > \Phi^{u_{01}}.$$

Finally, consider the h_1-network. From Proposition 9, $\Phi^{h_1} = \Phi^{u_{12}} > \Phi^{u_{01}}$ for $\mu_{12} \in [\overline{\kappa}, 4\mu]$. For $\mu_{12} < \overline{\kappa}$, we see from (32) and (36) that $\Phi^{h_1} > \Phi^{u_{12}}$ if and only if

$$Z := 4(N+1)^2[x^2\mu + 2(x+2)(x+1)\mu_{12}] - [4(N+1) + 3x(N-x)]^2\mu_{12} > 0.$$

Now

$$\begin{aligned} Z &= x^2\left[4(N+1)^2\mu - (N-x)^2\mu_{12}\right] \\ &\quad + 8\left[(N+1)^2(x+2)(x+1) - 2(N+1)^2 - x^2(N-x)^2 - 3x(N-x)(N+1)\right]\mu_{12}. \end{aligned}$$

The first term in this sum is positive since μ_{12} is at most 4μ. The second term is positive as well. Therefore $Z > 0$ as desired. As in the case of the unrestricted network, this also establishes that $\Phi^{h_1} > \Phi^{u_{01}}$ for $\mu_{12} \geq \mu$. For $\mu_{12} < \mu$, using (31) and (36):

$$\begin{aligned} \Phi^{h_1} &> 4\mu \cdot \frac{4(N-x+1)^2 - (N-x)(N-x+2)}{[4(N+1)+3x(N-x)]^2} \\ &= 4\mu \cdot \frac{3(N-x)(N-x+2)+4}{[4(N+1)+3x(N-x)]^2} \\ &> \Phi^{u_{01}}. \end{aligned}$$

Having established that unary networks are always dominated, we now compare the other networks. All the equalities in (i)–(vi) of the proposition follow immediately from Proposition 9. The inequalities in (i), (iii), and (vi) can be verified by direct computation. For (i), $\mu_{12} = 0$, we see from (30) and (32) that

$$\Phi^{h_0} = \frac{16\mu}{(3N+4)^2},$$
$$\Phi^{h_1} = \frac{4\mu}{(N+4)^2}.$$

It is easy to verify that $\Phi^{h_1} = \Phi$ if $N = 4$, $\Phi^{h_1} > \Phi$ if $N > 4$, and $\Phi^{h_1} < \Phi$ if $N < 4$. For (iv), $\mu_{12} = \mu$, we use (28) and (30) to obtain:

$$\Phi = \frac{4\mu}{(N+2)^2},$$
$$\Phi^{h_0} = \frac{16(3N^2 + 12N + 16)\mu}{(N+4)^2(3N+4)^2}.$$

After some algebraic manipulation we see that that $\Phi^{h_0} > \Phi$ provided $N \geq 5$, and $\Phi^{h_0} < \Phi$ if $N \leq 4$. For (vi), $\mu_{12} \in [\overline{\kappa}, 4\mu]$, the argument is in the text.

For (ii), (iii) and (v), we invoke an asymptotic argument. Let $\Phi_\infty := \lim_{N\to\infty} N^2\Phi$, and $\Phi_\infty^{h_k} := \lim_{N\to\infty} N^2\Phi^{h_k}$. Similarly define $x_\infty := \lim_{N\to\infty} \frac{x}{N}$, the limiting proportion of arbitrageurs on link (0,1). We are interested in x_∞ in the unrestricted network and in the h_1-network; in order to distinguish the two we index x_∞ by u or h. We already know the values of x_∞ in the benchmark cases. At $\mu_{12} = 0$, $x_{u,\infty} = \frac{1}{2}$ and $x_{h,\infty} = \frac{2}{3}$. At $\mu_{12} = 4\mu$, $x_{u,\infty} = x_{h,\infty} = 0$. Also, at $\mu_{12} = \mu$, $x_{u,\infty} = \frac{1}{3}$ and $x_{h,\infty} = \frac{1}{2}$. For the remainder of the proof we take $\mu \in (0, 4\mu)$. Then, all admissible links are active for sufficiently large N, so that equations (29) and (33) apply. Correspondingly, $x_{u,\infty}$ and $x_{h,\infty}$ solve:

$$9(\mu_{12} - \mu)x_{u,\infty}^2 + (3x_{u,\infty} - 1)(4\mu - \mu_{12}) = 0, \tag{37}$$
$$3(\mu_{12} - \mu)x_{h,\infty}^2 + (2x_{h,\infty} - 1)(4\mu - \mu_{12}) = 0. \tag{38}$$

Implicitly differentiating (37) with respect to μ_{12}, we get

$$\frac{dx_{u,\infty}}{d\mu_{12}} \cdot 3[4\mu - \mu_{12} + 6x_{u,\infty}(\mu_{12} - \mu)] + (3x_{u,\infty} - 1)^2 + 3x_{u,\infty} = 0.$$

The coefficient of $\frac{dx_{u,\infty}}{d\mu_{12}}$ is clearly positive for $\mu_{12} \geq \mu$. It is also positive for $\mu_{12} < \mu$ since $x_{u,\infty} < \frac{1}{2}$. Therefore $\frac{dx_{u,\infty}}{d\mu_{12}} < 0$. Similarly we can show that $\frac{dx_{h,\infty}}{d\mu_{12}} < 0$. Provided $\mu_{12} \neq \mu$, (37) and (38) give us:

$$\begin{aligned}(3x_{u,\infty} - 2x_{h,\infty})(4\mu - \mu_{12}) &= 3(\mu - \mu_{12})(3x_{u,\infty}^2 - x_{h,\infty}^2)\\ &= (\mu - \mu_{12})\left[3x_{u,\infty} + \sqrt{3}x_{h,\infty}\right]\\ &\quad \cdot \left[3x_{u,\infty} - 2x_{h,\infty} + \left(2 - \sqrt{3}\right)x_{h,\infty}\right],\end{aligned}$$

so that

$$(3x_{u,\infty} - 2x_{h,\infty})\left[\frac{4\mu - \mu_{12}}{\mu - \mu_{12}} - a\right] = a\left(2 - \sqrt{3}\right)x_{h,\infty}, \tag{39}$$

where

$$a := 3x_{u,\infty} + \sqrt{3}x_{h,\infty} < 3 \cdot \frac{1}{2} + \sqrt{3} \cdot \frac{2}{3} < 4.$$

The RHS of (39) is positive. For $\mu_{12} > \mu$ it is immediate that $x_{u,\infty} < \frac{2}{3}x_{h,\infty}$. For $\mu_{12} < \mu$, it is easy to check that $4\mu - \mu_{12} > 4(\mu - \mu_{12})$, so that in this case $x_{u,\infty} > \frac{2}{3}x_{h,\infty}$.

From (28), (30), (31), and (32):

$$\Phi_\infty = \frac{4\mu_{12}}{(2 - 3x_{u,\infty})^2}, \tag{40}$$

$$\Phi_\infty^{h_0} = \frac{16}{9}(\mu + 2\mu_{12}), \tag{41}$$

$$\Phi_\infty^{h_1} = \frac{4}{9} \cdot \frac{4\mu - \mu_{12}}{x_{h,\infty}^2}, \tag{42}$$

$$\Phi_\infty^{h_1} = \frac{4}{9} \cdot \frac{\mu + 2\mu_{12}}{(1 - x_{h,\infty})^2}. \tag{43}$$

Also, provided $\mu_{12} \neq \mu$, we have (using (37), (38), (40), and (42)):

$$\Phi_\infty = \frac{4}{3} \cdot \frac{\mu_{12} - \mu}{1 - 3x_{u,\infty}}, \tag{44}$$

$$\Phi_\infty^{h_1} = \frac{4}{3} \cdot \frac{\mu_{12} - \mu}{1 - 2x_{h,\infty}}. \tag{45}$$

Consider part (v) of the proposition. We have $\mu_{12} > \mu$, so that $x_{u,\infty} < \frac{2}{3}x_{h,\infty}$ and $x_{h,\infty} < \frac{1}{2}$. For fixed μ_{12}, the expression in (43) is strictly increasing in $x_{h,\infty}$. Therefore $\Phi_\infty^{h_1} < \frac{16}{9}(\mu + 2\mu_{12})$, i.e. $\Phi_\infty^{h_0} > \Phi_\infty^{h_1}$ from (41). From (44) and (45), $\Phi_\infty^{h_1} > \Phi_\infty$.

We now turn to parts (ii) and (iii) of the proposition, with $\mu_{12} < \mu$. In this case, $x_{u,\infty} > \frac{2}{3}x_{h,\infty}$ and $x_{h,\infty} > \frac{1}{2}$. By the same logic as above $\Phi_\infty^{h_1} > \frac{16}{9}(\mu + 2\mu_{12})$, so that $\Phi_\infty^{h_1} > \Phi_\infty^{h_0}$. From (44) and (45), $\Phi_\infty^{h_1} > \Phi_\infty$.

That $\Phi_\infty^{h_0} \geq \Phi_\infty$ is easy but tedious to verify. First, notice that with $\mu_{12} := \delta\mu < \mu$, the relevant quadratic implies that $x_{u,\infty} = \frac{(4-\delta)-\sqrt{3\delta(4-\delta)}}{6(1-\delta)}\mathbf{1}_{\delta>\delta^*} + \frac{1}{2}\mathbf{1}_{\delta\leq\delta^*}$, where $\delta^* := 1/7$ is the lowest δ so that the root of the quadratic is still in $[0, 1/2]$. For $\delta < \delta^*$, the root is strictly larger than $1/2$. Assume first that $\delta > \delta^*$. From equations (41) and (44), and inserting the expression for $x_{u,\infty}$, the inequality can be expanded to read $0 < f(\delta) := 27 + 128\delta + 42\delta^2 - 288\delta^3 + 91\delta^4$. With $f(0) = 9$ and $f(1) = 0$, and f first increasing and then decreasing on $(0, 1)$, the fact that f is first convex and then strictly concave on $(0, 1)$ implies that $f(\delta) > 0$ for $\delta \in (\delta^*, 1)$, as required for $\Phi_\infty^{h_0} > \Phi_\infty$. If on the other hand $\delta \leq \delta^*$, then we know that $x = 1/2$, and as a result $\Phi_\infty^{h_0} = \Phi_\infty$. ■

Proof of Proposition 11 For the $u_{k\ell}$-network, an equilibrium security design is given by Proposition 3. In particular, **S** holds, so that the no-default constraints of arbitrageurs are satisfied with equality. The egalitarian welfare function is $W := \sum_{j \in K} W^j = W^k + W^\ell$. Using, in sequence, (17), (4), and (6):

$$\begin{aligned} W &= \frac{1}{\beta^k}\|p^k - \hat{p}^k\|_2^2 + \frac{1}{\beta^\ell}\|p^\ell - \hat{p}^\ell\|_2^2 \\ &= \beta^k\|R^k y^k\|_2^2 + \beta^\ell\|R^\ell y^\ell\|_2^2 \\ &= (\beta^k + \beta^\ell)\|R^k y^k\|_2^2 \\ &= \frac{N^2}{\beta^k + \beta^\ell}\|\hat{p}^k - \hat{p}^\ell\|_2^2 \end{aligned}$$

which is proportional to arbitrageur profits (9). Hence the profit-maximizing unary network also maximizes W. ■

References

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Culbertson, J. (1957). The term-structure of interest rates. *Quarterly Journal of Economics*, 71:485–517.

Duffie, D. and Jackson, M. (1989). Optimal innovation of futures contracts. *Review of Financial Studies*, 2(3):275–296.

Gabszewicz, J. J. and Vial, J.-P. (1972). Oligopoly 'a la cournot' in a general equilibrium analysis. *Journal of Economic Theory*, 4:381–400.

Goyal, S. and Joshi, S. (2005). Unequal connections. Mimeo, University of Essex.

Jackson, M. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71:44–74.

Mas-Colell, A. (1982). The cournotian foundations of walrasian equilibrium theory: an exposition of recent theory. In Hildenbrand, W., editor, *Advances in Economic Theory*. Cambridge University Press.

Modigliani, F. and Sutch, R. (1966). Innovations in interest rate policy. *American Economic Review*, 56:178–197.

Nikaido, H. (1968). *Convex Structures and Economic Theory*, volume 51 of *Mathematics in Science and Engineering*. Academic Press.

Rahi, R. and Zigrand, J.-P. (2005). Financial innovation in segmented markets. Mimeo, The London School of Economics.

Takayama, A. (1985). *Mathematical Economics, Second Edition*. Cambridge University Press.

Tufano, P. (1989). Financial innovation and first-mover advantages. *Journal of Financial Economics*, 25:213–240.